ESSAYS ON SPECIFICATION AND ESTIMATION OF MODELS OF MARKETS FOR HETEROGENEOUS HOUSING

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A dissertation submitted to the graduate school in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in ECONOMICS AND PUBLIC POLICY

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INTRODUCTION

Housing markets are central to the economy. Housing provides a fundamental need of all households and, for many of them, serves as their largest investment. My dissertation examines housing markets, and proposes new ways to understand them through models that capture some of their particularities. My approach treats an entire metropolitan area as a housing market. A central challenge in modeling these housing markets is the heterogeneity and unobservability of housing quality and the associated difficulty of separating quantity from price. My dissertation proposes a new approach to modeling heterogeneity both in quality of housing and preferences of households.

Chapter 1 presents a framework that characterizes the determination of the price of housing as a function of quality for the entire housing distribution while treating quality as unobservable. By its nature, house quality is not observed by the econometrician. One approach, from the hedonic literature, is to estimate a mapping from the observed characteristics to the house value. This approach assumes that housing can be characterized by a vector of characteristics, each having some well defined cardinality, and that unobserved heterogeneity is not systematically related to observables. However, measuring housing characteristics is in practice challenging, and strong assumptions are required for identification and estimation. My model bypasses this step and provides a new method for estimating the price-quality frontier for housing, treating housing quality as unobserved by the econometrician. Heterogeneity among housing units is captured by a one dimensional (latent) index. One of the key insights in Rosen [1974] is that the equilibrium pricing function can be characterized by the solution to a nonlinear differential equation. This insight is followed with a flexible parameterization of the model that yields a tractable closed form solution to the equilibrium price function. Identification relies on multiple cross-sections of data for a given metropolitan area. The model characterizes the way in which the number of households, and the distribution of characteristics (income and family size) across households, gives rise to a distribution of demand for housing quality. The distribution of quality is fixed at a given point in time, and this distribution, coupled with the distribution of demand, gives rise to an equilibrium price function, called a hedonic price function, determining price as a function of quality for the entire distribution of houses in the market. The framework also characterizes the change in supply of each quality from period to period as a function of the change in the equilibrium asset value of each quality type. This framework is then used to gain insight into the determinants of the change in houses of various quality types from period to period. This model give a theoretical connection between
fundamentals (income distribution) and prices observed in the housing market. With houses defined as assets whose value is determined by a flow of rents through time, this model can be used to gain insight into the causes and effects of the recent housing market crisis in the U.S, in particular into why such a large run-up and subsequent collapse in values was observed while similar behavior was not present in the income distributions.

Chapter 2 presents an application of this model to study the housing markets using data from the American Housing Survey before and during the recent housing market crisis in the US. The application focuses on the housing markets of Miami (FL) which experienced an average real appreciation of housing values of 65 percent during the period from 2002 to 2007. As a shorthand, the common parlance of terming this as bubble is adopted without endowing this term with any connotations as to whether investor behavior was or was not rational. Estimation of the model using three successive cross-sections (1995, 2002, 2007) of data for Miami reveals that the model fits well. The results provides new insights into the causes and effects of the dramatic run up in housing prices that occurred during the period leading up to the recent recession in the U.S. Changes in real income, housing supply and population growth can only account for a small fraction of the observed changes in housing values between 2002 and 2007 (bubble period). A comparison is made between the observed prices and the prices implied by the model for this period to quantify the bubble. When looking for a possible cause of this run-up, I find that interest and depreciation rates did not change much during that time period either. The model accounts for these changes in housing values as arising from a change in investor expectations regarding future appreciation in rental value of housing. A high expected rate of appreciation resulted in a drop of the user cost factor of almost 50 percent from the pre-bubble level. The subsequent fall in housing prices indicates that expectations driving this fall in user cost were not realized.

Heterogeneity in tastes plays an important role in the behavior of households in the housing market. Sorting on income is an important characteristic of equilibrium sorting models, a strand of literature to which this model belongs. When sorting on income is contrasted against the data in the housing market, it is not very closely satisfied. However, when households are divided by simple demographic characteristics I observe that it is significantly better satisfied in the data, suggesting an important heterogeneity in preferences that is linked to these demographic differences. Going beyond a single type of household presents additional challenges for estimation and identification. Chapter 3, provides a new method for estimating the price-quality frontier of housing with heterogeneity in households preferences.

This model discretizes the pricing function for quality. Rather than a closed form solution, a numerical solution to the differential equation characterizing the hedonic price function is obtained. This provides greater precision in fitting the data. I do not require to discretize the income distribution. The model in chapter 1 is extended by al-
lowing for variation in household characteristics, such as household size, in addition to variation in income. Identification and estimation of the model is explained. While sharing some characteristics to the treatment of chapter 1, the current approach presents alternative advantageous features over that one. First, the current model allows for flexibility of utility functional forms. Second, a new solution and estimation algorithm is developed for this generalized class of models that is based on discrete approximations of the distribution of housing quality. Third, no assumptions on the form of the distributions of exogenous or endogenous variables is required. Fourth, the inclusion of several types allows for additional sources of observed heterogeneity, which allows the model to be used for the evaluation of welfare effect of policies with differential impact across demographic groups.

As empirical applications, I explore alternative criteria for the separation of households into subgroups to identify key demographic characteristics that, in addition to income, give rise to differences in preferences with respect to housing. The first application of our model uses k-means clustering to identify household types in 2 dimensions, age of householder and number of children, according to their behavior in the market. This data mining technique allows us to learn structure from the data as an input for our structural model. I find 3 clusters: Very young households with few or no children, middle age households with children, and older households with no children (living in the household). In second place, I present a full estimation of the model defining types as households with and without children. This division satisfies our idea of types because the presence of children is often a decisive variable when choosing a housing unit. I estimate the model using periodic American Housing Surveys for 1999 and 2003 for New York City. Then, I include New York City and Chicago in a joint estimation. This will allow comparisons between distribution of housing qualities and the associated prices across metropolitan areas. Analysis of the differences in preferences allows to conclude, for example, that the presence of children lowers the preference for housing quality relative to other consumptions, which is accompanied by an increase in income and price elasticities.

One natural generalization is the incorporation of multiple metropolitan areas. Chapter 4 develops a model that allows for the inclusion of multiple metropolitan areas. I exploit similar assumptions to those in the single metropolitan area to obtain identification of the preference parameters and the pricing functions. In particular, I extend the time invariance assumption for preferences and assume preferences are invariant across metropolitan areas as well. I estimate this extended model for two major metropolitan areas, Chicago and New York, in 1999 and 2003. The results provide a comparison of the quantity of housing of each quality for the two metropolitan areas. I estimate the compensating variation arising from the difference in quality and price a household of a given income would obtain in a larger relative to a smaller metropolitan area. This welfare measure is of interest in its own right and provides valuable insights into agglomeration eco-
nomics. In particular, the aggregate of the compensating variations across the entire set of households occupying the larger metropolitan area provides a measure of the aggregate compensating variation foregone by households choosing to reside in the larger metropolitan area, and hence a measure of the minimum agglomeration benefits that must be provided by the larger metropolitan area. For instance, I find that for a household at the 50th income percentile in Chicago, a CV of approximately 20% of income is required to induce to move that household to New York.
Part I

HOUSING MODELS WITH HOMOGENEOUS HOUSEHOLD PREFERENCES FOR QUALITY
A MODEL OF THE PRICE-QUALITY FRONTIER FOR HOUSING WITH LATENT HETEROGENEOUS QUALITY.

1.1 INTRODUCTION

A central challenge in estimating models with heterogeneous housing is separating quality from price. By its nature, house quality is not directly observed by the econometrician. One approach, from the hedonic literature, is to estimate a mapping from the observed characteristics of a house to its value. This approach assumes that housing can be characterized by a vector of characteristics and that unobserved heterogeneity is not systematically related to observables.\(^1\) Measuring housing characteristics is in practice challenging and creates a variety of well-known identification and estimation problems due the potential of omitted variables. I bypass this step and provide a new method for estimating hedonic price functions treating housing quality as unobserved by the econometrician.

The pioneering work of Rosen [1974] transformed modeling of markets for differentiated products.\(^2\) A great many fruitful applications have built on the Rosen’s framework. I advance this agenda by developing a framework for estimating hedonic equilibrium models of metropolitan-wide housing markets. I distinguish between rental and asset markets for housing and show how to characterize and estimate both nonlinear equilibrium pricing functions. Key elements of our approach are the following. I threat the metropolitan area as a unified housing market. I show that rental and asset prices as a function of quality are separately identified for the entire distribution of housing in the metropolitan area, yielding an estimator of the quality distribution of housing in the entire market. I define quality in the broadest terms to incorporate not only structural housing characteristics per se, but also all associated natural and publicly provided amenities. Quality is latent so that it is not necessary to differentiate between observed and unobserved housing characteristics. Finally, I also show how to estimate period-to-period changes in housing supply as a function of quality.

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1 This has been an important agenda of hedonic theory and of associated empirical work linking house values to observed house characteristics. For a review of the recent literature see, for example, Kuminoff et al. [2013].

2 Closely related to the hedonic literature is also the work by Berry and Pakes [2007] on the pure characteristics model.
Implementation of the approach is feasible with readily available data for metropolitan housing markets in the U.S. Our key simplifying assumption is that quality can be mapped onto a unidimensional index. In this respect, our approach is positioned between two widely employed characterizations of housing markets. One treats housing as a homogeneous and perfectly divisible commodity. The other treats housing as comprised of a fixed stock of heterogeneous housing types. Our approach occupies the middle ground, with housing being continuous and unidimensional, as in the former approach, while being heterogeneous along the quality dimension and inelastically supplied within-period, as in the latter. In adopting this unidimensional characterization of housing, I forgo the potential of the multi-attribute framework for valuing individual elements of the bundle of housing attributes. I also avoid the severe challenges entailed in extending the multi-attribute framework to a multi-period setting; modeling changes in the stock of houses with varying attribute bundles and the associated equilibrium implicit prices of the attributes is difficult if not intractable. In exchange for our simplifying assumption, I gain a tractable framework that is ideally suited to study the distributions of quality and price in metropolitan housing markets and associated changes over time while exploiting the full set of equilibrium conditions at each point in time.

I show that there exists a new flexible parametrization of Rosen’s model that exploits generalized forms of the log-normal distributions Vianelli [1983] and that yields tractable solutions for the equilibrium pricing function. When quality is latent there is an obvious identification problem since there is no inherent scale for housing quality. For every non-linear pricing model, there exists a transformation of the utility function such that this transformation is observationally equivalent to the original model and pricing is linear. I can, therefore, normalize housing quality by setting it equal to the rental price in a baseline period and identify preferences for housing from the observed income expansion paths. I need data for more than one time period or multiple spatial markets to identify non-linearities in pricing of housing.

In addition, I must overcome one more identification problem, that is common in this literature, but has not received proper attention. For rental units, I observe rental prices, but not housing values. For owner-occupied units, I observe housing values, but not rental prices. As a consequence, both rental rates and values are only partially observed by the econometrician. This imposes the need to identify the equilibrium rent-to-value ratio as a function of quality. I consider investors who trade real estate assets and model the equilibrium in these competitive asset markets. I show that the value of the house is

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3 The former dates to the classic works of Muth [1960, 1969] and Mills [1972] while the later was pioneered by Dunz [1989] and Nechyba [1997, 2000].

4 Exploiting variations among multiple markets is also a useful strategy to obtain identification if characteristics are observed as discussed in Bartik [1987] and Epple [1987]. Alternative strategies for identification are discussed in Ekeland et al. [2004], Bajari and Benkard [2005], and Heckman et al. [2010] and P. et al. [2012].
given by the expected net present value of the discounted stream of rental income. Housing values and rents are, therefore, closely linked in equilibrium. One cannot analyze values separately from rents. At each point of time, the proportionality between rents and values can be captured by a time varying quality dependent rent-to-value function. I show that these functions are non-parametrically identified by characterizing the set of households that are indifferent between owning and renting for a given level of housing quality.

Our econometric findings provide valuable new evidence about metropolitan housing markets. I provide two applications. I study changes in price across the quality distribution in a metropolitan area (Miami) that experienced dramatic housing prices during the recent housing bubble. I find housing prices relative to annualized rents increased over the entire quality spectrum, but with especially pronounced increases at the lower end of qualities. These findings accord well with the widespread reporting of eased access to credit, especially for lower-income buyers, during the housing price run-up.5

In our second application, I estimate the model simultaneously for two different large metropolitan areas, New York and Chicago. The results provide a comparison of the quantity of housing of each quality for the two metropolitan areas. I estimate the compensating variation arising from the difference in quality and price a household of a given income would obtain in a larger relative to a smaller metropolitan area. This welfare measure is of interest in its own right and provides valuable insights into agglomeration economics. In particular, the aggregate of the compensating variations across the entire set of households occupying the larger metropolitan area provides a measure of the aggregate compensating variation foregone by households choosing to reside in the larger metropolitan area, and hence a measure of the minimum agglomeration benefits that must be provided by the larger metropolitan area. I find that for a household at the 50th income percentile in Chicago, a CV of approximately 20% of income is required to induce to move that household to New York.

1.2 AN EQUILIBRIUM MODEL OF HOUSING

I consider the determination of asset prices and rental rates for houses with heterogeneous quality. Our model distinguishes between housing services, defined as the period flow of housing consumption, and housing assets. Housing values or prices for real estate assets depend on prevailing and expectations about future interest rates, costs of homeownership, property taxes and rental rates for housing services. Real estate values are determined in asset markets. Housing services

5 Our modeling approach is a closely related to recent work on nonlinear pricing in housing markets by Landvoigt et al. [2011] who find that the relaxation of credit constraints had a large impact on house prices in San Diego. Their findings are broadly consistent with our findings. While their results are based on a more sophisticated dynamic model than the one I employ, our approach focuses on empirical tractability based on formally derived identification results and feasible semi-parametric estimation. In that sense both approaches are complimentary.
can be rented in frictionless markets that allow for nonlinear pricing of housing quality. Current and future rental rates partially determine housing values, while housing values partially determine the supply of new housing units. As a consequence, both markets cannot be studied in isolation.

1.2.1 Asset Markets

First, I consider the asset markets for housing. Housing units differ by quality, which can be characterized by a one-dimensional ordinal measure denoted by $h$. For each level of housing quality $h$, there is an asset market in which investors can buy and sell houses at the beginning of each period. Let $V_t(h)$ denote the asset price of a house of quality $h$ at time $t$.

**Assumption 1** Private investors are risk neutral.

Note that I can weaken this assumption. I only need to require that the marginal investor is risk neutral. Investors can borrow capital in short term bond markets. The one-period interest rate is denoted by $i_t$. Investors are also responsible for paying property taxes to the city. The property tax rate is given by $\tau^p_t$. Finally owners have additional costs due to appreciation and maintenance that occurs with rate $\delta_t$.

**Assumption 2** Asset markets are competitive.

The expected profits, $\Pi_t$, of buying a house with quality $h$ at the beginning of period $t$ and selling it at the beginning of the next period is then given by:

$$E_t[\Pi_t(h)] = E_t \left[ -V_t(h) + v_t(h) + \frac{V_{t+1}(h)(1 - \tau^p_{t+1} - \delta_{t+1})}{1 + i_t} \right]$$

(1)

where the first term reflects the initial investment, the second term the flow profits from rental income at time $t$, and the last term the discounted expected value of selling the asset in the next period.\(^6\)

Since investors are risk neutral and entry into the profession is free, expected profits for investors must be equal to zero. Hence housing values or asset prices must satisfy the following no-arbitrage condition:

$$0 = E_t \left[ -V_t(h) + v_t(h) + \frac{V_{t+1}(h)(1 - \tau^p_{t+1} - \delta_{t+1})}{1 + i_t} \right]$$

(2)

Solving for $V_t(h)$, I obtain the following recursive representation of the asset value at time $t$:

$$V_t(h) = v_t(h) + \frac{(1 - \tau^p_{t+1} - \delta_{t+1})}{(1 + i_t)} E_t [V_{t+1}(h)]$$

\(^6\) For analytical convenience, I are assuming that property taxes and maintenance expenditures are due at the beginning of the next period.
By successive forward substitution of the preceding, I obtain:

\[ V_t(h) = v_t(h) + E_t \sum_{j=1}^{\infty} \beta_{t+j} v_{t+j}(h) \]  

(3)

where the stochastic discount factor is given by:

\[ \beta_{t+j} = \prod_{k=1}^{j} \frac{(1 - \tau_{t+k} - \delta_{t+k})}{(1 + i_{t+k-1})} \]  

(4)

This demonstrates that the asset value of a house of quality \( h \) is the expected discounted flow of future rental income. The discount factors \( \beta_{t+j} \) depend on interest rates, property tax rates and depreciation rates. An alternative instructive way of writing this expression is as follows.

**Proposition 1** Let \( 1 + \pi_t(h) = \frac{v_{t+j}(h)}{v_{t+j-1}(h)} \) denote the rate of housing inflation at date \( t \). Define \( \tilde{\beta}_{t+j} \) as follows:

\[ \tilde{\beta}_{t+j}(h) = \prod_{k=1}^{j} \frac{(1 - \tau^p_{t+k} - \delta_{t+k}) (1 + \pi_{t+k}(h))}{(1 + i_{t+k-1})} \]  

(5)

Then:

\[ V_t(h) = \frac{v_t(h)}{c_t(h)} \]  

(6)

where \( c_t(h) \) is the user cost of capital:

\[ c_t(h) = \frac{1}{1 + E_t \sum_{j=1}^{\infty} \tilde{\beta}_{t+j}(h)} \]  

(7)

Consider the time-invariant case studied by Poterba [1984, 1992]:

\[ E_t \prod_{k=1}^{j} \frac{(1 - \tau^p_{t+k} - \delta_{t+k})(1 + \pi_{t+k}(h))}{(1 + i_{t+k-1})} = \left[ \frac{(1 - \tau^p - \delta)(1 + \pi(h))}{1 + i} \right]^j \]  

(8)

When \( \tau^p, \delta, \pi, \) and \( i \) are small, the preceding closely approximates the continuous time solution of Poterba [1984]: \( u(h) = (i + \tau^p + \delta - \pi(h)) \).

Our model does not necessarily assume that investors have correct expectations about housing rental appreciation. It is possible that expectations of rental price increases prove to be greater than the actual rates of increase that are realized.

### 1.2.2 Rental Markets

To complete our model of asset prices, I need to derive the equilibrium rent function that prevails in the market for housing services. One could follow a number of different approaches to derive this function. Here I follow the hedonic literature and allow for non-linear pricing in a rental market for housing services. There is a continuum of renters with mass equal to \( N_t \). I normalize the population at the
initial date to be one \((N_1 = 1)\) and treat \(\{N_t\}_{t=1}^{\infty}\) as an exogenous process.

Renters differ in income denoted by \(y\).\(^7\) Let \(F_t(y)\) be the metropolitan income distribution at time \(t\). Renters have preferences defined over housing services \(h\) and a composite good \(b\). Let \(U_t(h, b)\) be the utility of a household at time \(t\).

Since housing quality is ordinal, housing quality is only defined up to a monotonic transformation. Given such a normalization, I can define a mapping \(v_t(h)\) that denotes the period \(t\) rental price of a house that provides quality \(h\). The transactions cost in the rental market are zero. Hence, renters can costlessly change its housing consumption on a period-to-period basis as rental rates change. It follows that a renter’s optimal choice of housing to rent at each date \(t\) maximizes its period utility at date \(t\):

\[
\max_{h_t, b_t} U_t(h_t, c_t) \quad (9)
\]

s.t. \(y_t = v_t(h_t) + c_t\)

where \(c_t\) denotes expenditures on a composite good.

The first-order condition for the optimal choice of housing consumption is:

\[
m_t(h_t, y_t - v_t) \equiv \frac{U_h(h_t, y_t - v_t)}{U_c(h_t, y_t - v_t)} = v'_t(h_t) \quad (10)
\]

Solving this expression yields the housing demand \(h_t(y_t, v_t(h))\). Integrating over the income distribution yields the aggregate housing demand \(H_t^d(h|v_t(h))\):

\[
H_t^d(h|v_t(h)) = \int_0^\infty 1\{h_t(y, v_t(h)) \leq h\} \, dF_t(y) \quad (11)
\]

where \(1\{\cdot\}\) denotes an indicator function. Thus \(H_t^d(h|v_t(h))\) is the fraction of renters whose housing demand is less than or equal to \(h\).

To characterize household sorting in equilibrium, I impose an additional restriction on household preferences.

**Assumption 3** The utility function satisfies the following single-crossing condition:

\[
\frac{\partial m_t}{\partial y} \bigg|_{U_t(h, y - v(h)) = \bar{U}} > 0 \quad (12)
\]

Assumption 1 states that high-income renters are willing to pay more for a higher quality house than low-income renters – a weak restriction on preferences. The single-crossing condition implies the following result.

---

\(^7\) I can extend this model and allow for additional sources of heterogeneity as discussed in Quintero (2013). The main advantage of our approach here is that I can obtain closed form solutions to the equilibrium rental price function, as I will see below, which is helpful to establish the basic identification results. All of the key results go through in more general class of models discussed in Quintero [2014].
Proposition 2 If \( F_t(y) \) is strictly monotonic, then there exists a monotonically increasing function \( y_t(v) \) which is defined as

\[
y_t(v) = F_t^{-1}(G_t(v))
\] (13)

Note that \( y_t(v) \) fully characterizes household sorting in equilibrium.

1.2.3 Population Growth and Changing Supply

Thus far our model has treated the population and the distribution of housing quality as fixed. I can extend the model to accommodate population and housing supply change. Let \( N_t \) denote the metropolitan population at date \( t \). Normalize the population at the initial date to be one: \( N_1 = 1 \). Let \( q_t(h) \) denote the quantity of housing of quality \( h \) at date \( t \). Let the housing supply function for quality \( h \) be:

\[
q_t(h) = s(q_{t-1}(h), V_t(h), V_{t-1}(h))
\] (14)

Supply of quality \( h \) at date \( t \) thus depends on the quantity of that housing quality the previous period, the values of houses of that quality in the previous and current periods. This formulation reflects the fact that home builders produce and sell dwellings and hence are concerned about the market value of the dwelling, \( V_t(h) \), and not implicit rent. Including lagged values of quantity and price serves to capture potential adjustment costs.

Assumption 4 I adopt the following constant-elasticity parametric form for this supply function:

\[
q_t(h) = q_{t-1}(h) \left( \frac{V_t(h)}{V_{t-1}(h)} \right) ^{\zeta}
\] (15)

While this function is not explicitly derived from specification of cost function for the producer, it has attractive properties. It is parsimonious; it introduces only one additional parameter, \( \zeta \). Equation (83) also implies that the stock of housing of quality \( h \) does not change from date \( t - 1 \) to date \( t \) if the the rental price of that quality of housing does not change. If the rental price of housing type \( h \) rises, the quantity rises as a constant elasticity function of the proportion by which the price increases. If the price of housing type \( h \) falls, the quantity declines reflecting depreciation and reduced incentive to invest in maintaining the housing stock. The magnitude of the response depends on the elasticity \( \zeta \). Hence, our model of unchanging supply corresponds to \( \zeta = 0 \).

In period one, I take the housing stock, \( R_1(h) \), as given. The market clearing condition for the housing market in period one is then:

\[
G_1(v_1(h)) = R_1(h)
\] (16)

The first-period population is normalized to one, implying

\[
R_1'(h) = q_1(h)
\] (17)
Consider periods $t > 1$. Market clearing requires that prices and quantities adjust so that everyone is housed:

$$N_t = \int_0^\infty q_t(h) \, dh = \int_0^\infty q_{t-1}(h) \left( \frac{V_t(h)}{V_{t-1}(h)} \right)^\zeta \, dh$$  \hspace{1cm} (18)

The distribution of housing supply in period $t$ is:

$$R_t(h) = \frac{1}{N_t} \int_0^h q_t(x) \, dx = \frac{1}{N_t} \int_0^h q_{t-1}(x) \left( \frac{V_t(x)}{V_{t-1}(x)} \right)^\zeta \, dx$$  \hspace{1cm} (19)

I thus obtain a recursive relationship governing the evolution of the supply of housing over time. Market clearing in the housing market at date $t$ requires:

$$G_t(v_t(h)) = R_t(h) = \frac{1}{N_t} \int_0^h q_{t-1}(x) \left( \frac{V_t(x)}{V_{t-1}(x)} \right)^\zeta \, dx$$  \hspace{1cm} (20)

The expressions for the remainder of the model are unchanged. The "number" of households of income $y$ at date $t$ is given by:

$$n_t^y(y) = N_t f_t(y)$$  \hspace{1cm} (21)

Similarly, the number of houses at rental $v$ is:

$$n_t^v(v) = N_t g_t(v)$$  \hspace{1cm} (22)

Single-crossing implies that, in equilibrium, the house rental expenditure at date $t$ by income $y$ must satisfy:

$$N_t F_t(y) = N_t G_t(v)$$  \hspace{1cm} (23)

or $F_t(y) = G_t(v)$. 

1.3 Equilibrium

In equilibrium rental markets must clear for each value of $h$. I can define an equilibrium in the rental market for each point of time as follows:

**Definition 1** A hedonic housing market equilibrium is an allocation of housing consumption for each renter and price function $v_t(h)$ such that

a) Renter behave optimally given the price function;

b) Housing markets clear, i.e., for each level of housing quality $h$, I have:

$$H_t^d(h | v_t(h)) = R_t(h)$$  \hspace{1cm} (24)

An equilibrium exists under standard assumptions discussed in the hedonic literature.

To obtain a closed form solution for the equilibrium pricing function, I impose additional functional form assumptions.
Assumption 5 Income and housing are distributed generalized log-normal with location parameter (GLN4). 8

\[
\begin{aligned}
\ln(y_t) &\sim \text{GLN}_4(\mu_t, \sigma_t, \beta_t) \\
\ln(v_t) &\sim \text{GLN}_4(\omega_t, \tau_t, \theta_t)
\end{aligned}
\] (25)

I will show below that these functions are sufficiently flexible to fit the housing value and income distributions in all metro areas and all time periods that I consider in the empirical analysis.

Imposing the restriction that \( r_t = m_t \) permits us to obtain a closed-form mapping from house value to income. I then establish that the further assumption that \( \theta_t - \beta_t \) is time invariant permits us to obtain a closed-form solution to the hedonic price function. 9

Proposition 3 If \( r_t = m_t \ \forall t \), the income housing value locus is given by the following expression:

\[
y_t = \Lambda_t (v_t + \theta_t)^b_t - \beta_t
\] (26)

with \( a_t = \mu_t - \frac{\sigma_t}{\tau_t} \omega_t, \Lambda_t = e^{a_t}, \) and \( b_t = \frac{\sigma_t}{\tau_t}. \)

For our discussion of identification below, it is useful to note that all of parameters of the sorting locus, \( a_t = \mu_t - \frac{\sigma_t}{\tau_t} \omega_t, \Lambda_t = e^{a_t}, b_t = \frac{\sigma_t}{\tau_t}, \) and \( \theta_t \) can be estimated directly from the data. In addition, it will be useful below to note that if \( b_t > 1 \), this function is convex.

To obtain a closed form solution for the equilibrium price function, I adopt the following functional form for household preferences. 10

Assumption 6 Let utility given by:

\[
U = c_t(h) + \frac{1}{\alpha} \ln(y_t - v_t(h) - \kappa)
\] (27)

with \( c_t(h) = \ln(1 - \phi(h + \eta)^\gamma) \), where \( \alpha > 0, \gamma < 0, \phi > 0, \) and \( \eta > 0. \)

In addition to yielding a closed-form solution for the hedonic price function, this utility function proves to be relatively flexible in allowing variation in price and income elasticities. The conventionally defined income and price elasticities are obtained when the hedonic function is linear, i.e., when \( v(h) = ph \). The price elasticity of demand is then given by:

\[
\frac{dh}{dp} h = \frac{\left(-\alpha \phi \gamma h + (h + \eta)(h + \eta)^{-\gamma} - \phi\right)}{\left(-\alpha \phi \gamma + (1 - \gamma)(h + \eta)^{-\gamma} - \phi\right)} \frac{1}{h}
\] (28)

and the income elasticity of demand is given by:

\[
\frac{dh}{dy} h = \frac{-\alpha \phi \gamma}{p} \left[-\alpha \phi \gamma h + (h + \eta)(h + \eta)^{1-\gamma} - \phi(h + \eta)\right] \frac{1}{h} - \frac{\kappa}{\alpha \phi \gamma} \left[-\alpha \phi \gamma h + (h + \eta)^{1-\gamma} - \phi(h + \eta)\right]
\]

8 The four-parameter distribution for income simplifies to the standard two-parameter lognormal when the location parameter, \( \beta_t \), equals zero and the parameter \( r_t = 2. \) Similarly for the house value distribution. See Appendix A.2
9 I impose both of these restrictions when estimating our model.
10 This utility function requires the following two assumptions be satisfied \( 1 - \phi(h + \eta)^\gamma > 0 \) and \( y_t - v_t - \kappa > 0. \)
I will show that this specification of household preferences yields plausible price and income elasticities.

Given this parametric specification of the utility function, I have the following result:

**Proposition 4** If $b_t > 1$ ($\sigma_t > \tau_t$) and $\kappa = \theta_t - \beta_t \forall t$, the hedonic pricing function is well defined and given by:

$$v_t(h) = \left( A_t \left[ 1 - (1 - \phi(h + \eta)^\gamma)^{\alpha(b_t-1)} \right] \right)^{\frac{1}{1-b_t}} - \theta_t$$

for all $h > \left( \frac{1}{\phi} \right)^\gamma - \eta$

Note that $\frac{\sigma_t}{\tau_t} > 0$ is required for the price function to be increasing with $h$.

Our analysis of rental markets, therefore, provides an analytical characterization of the rental price of housing, $v_t(h)$, as a function of house quality, $h$. The market fundamentals determining $v_t(h)$ are the quality of the housing stock and the demand for housing services arising from the distribution of income in the metropolitan population. Equilibrium also depends indirectly on the equilibrium in asset markets since supply depends on asset values.

Finally, I should point out that this model can be extended to allow for a richer set of heterogeneity among agents. Using a discretized version of this model, Quintero (2014) considers models in which households also differ by other characteristics such as age, household size or children. Most of the results derived in this paper can be extended to this generalized framework although one loose the clean analytical solutions of pricing functions derived in this paper.

### 1.4 Identification

I consider identification of the model assuming that a) I have access to data for one market; and b) $h$ is not observed. Since housing quality is ordinal and latent, there is no well-defined unit of measurement for housing quality. This imposes some obvious limits on identification, as formalized by the following proposition.

**Proposition 5** For every model with equilibrium rental price function $v(h)$, there exists a monotonic transformation of $h$ denoted by $h^*$ such that the resulting equilibrium pricing function is linear in $h^*$, i.e. $v(h^*) = h^*$.

I can use arbitrary monotonic transformations of $h$ and redefine the utility function accordingly. Proposition 4 then implies that if I only observe data in one housing market and one time period, I cannot identify $u_t(h)$ separately from $v(h)$. A corollary of Proposition 4 is that I can normalize housing quality by setting $h = v_t(h)$ in some baseline period $t$. As I show in the proof of Proposition 5, this allows us to establish identification of the preference parameters. If, in addition, I make the standard assumption that per-period preferences are not changing over time, I can also identify the price functions in all subsequent periods.
Assumption 7 The utility function is time invariant.

Assumption 20 implies that the following result.

Proposition 6 The parameters of our utility function and the price function in all periods $t + s, s > 1$ are identified.

Once I normalize quality in the baseline period, the parameters of the utility function are identified of the observed income-expansion paths in the baseline period. Conditional on knowing the utility function, the rental price functions in all subsequent periods only depend on the observed joint distribution of rents and income. The proof of Proposition 5 provided in the appendix formalizes this result.

Thus far I have implicitly assumed that the distribution of rents is observed by the econometrician. Here, I discuss how to relax this assumption and account for the fact that rents are not observed for owner-occupied housing and need to be imputed.

One key implicit assumption of our model is that households are indifferent between renting from themselves (i.e. living in owner occupied housing) or from a third person. Households consumption is only a function of income holding prices fixed. Hence households with income $y$ consume the same number of housing independently whether they live in a rental unit, for which observe, $v_t(y)$, or live in an owner-occupied unit, for which I observe $V_t(y)$. By varying income $y$ I can trace out the equilibrium locus $V_t(v)$. As a consequence, I have the following result:

Proposition 7 There exists an equilibrium locus $v_t = v_t(V_t)$ which characterizes the rent of any housing unit as a function of its asset price. Moreover, this function is non-parametrically identified.

Proposition 6 implies that I can impute rents for owner-occupied using the rent-value functions. In practice, our data are more noisy since rents and values are not perfectly correlated with income as predicted by our model. However, I can use $E[v_t|y]$ and $E[V_t|y]$ to estimate the two sorting loci, $v_t(y)$ and $V_t(y)$, and proceed as discussed above. As a consequence the rent-to-value function is non-parametrically identified.

Our approach for identifying rent-to-value functions generalizes to models in which households are characterized by an observed vector of characteristics. The key assumption is that the average quality of housing consumption conditional on observed characteristics is the same for owners and renters, i.e. there is no sorting on unobservedables into home ownership. The non-parametric matching algorithm extends to more general demand models in which demands depends on a vector of observed state variables.

---

11 Quintero (2014) implements this extended approach. Overall, his findings are qualitatively and quantitatively similar to the results reported in this paper.

12 An alternative strategy to identify and estimate the rent-to-value function is discussed in Bracke [2013], who uses observations of houses that were both rented and sold within a short period.
Once I have identified the rent-to-value function, it is also straightforward to identify the housing supply function based on the market-clearing condition in periods $t \geq 2$. I have the following result.

**Proposition 8** The parameters of housing supply function are identified if I observe the equilibrium for, at least, two periods.

### 1.5 Estimation

The proofs of identification are constructive and can be used to define a three-step estimator for our model. First, I estimate the rent-to-value functions using a non-parametric matching estimator. I estimate the rent-value function for each time period allowing for changes in the user-cost functions across time periods. This approach captures changes in credit market conditions and investor expectations in a flexible non-parametric way. Second, I impute rents for owner-occupied housing and estimate the joint aggregate distribution of rents and income for each time period. Third, I estimate the structural parameters of the rental model using an extremum estimator which matches quantiles of the income and value distributions while imposing the parameter constraints in Propositions 23 and 24 and the housing market equilibrium restriction that $R_{t+s}(h) = G_{t+s}(v_{t+s}(h))$ for $j \geq 1$.

#### 1.5.1 Imputation of Rents for Owner Occupied Housing

In the previous analysis, I have implicitly assumed that the distribution of rents is observed by the econometrician. Here, I discuss how to relax this assumption and account for the fact that rents are not observed for owner-occupied housing and need to be imputed.

While I observe rents for rental properties, I need to impute rents for owner-occupied houses. Equation (91) can be used to impute rents for owner occupied housing. Solving the equation above for rents, I obtain

$$v_t(h) = u_t(h) V_t(h)$$

where $u_t(h)$ is the "user cost" of capital. Identification requires us to normalize $u_t$ for the baseline period. I can then impute the rents for owner occupied housing as a function of $u_{t+s}$ for $s > 1$. I then merge the observed distribution of rent for rental properties with the imputed distribution of rents for owner-occupied housing to obtain the aggregate distribution of rents. I treat $u(h) t + s$ as a parameter to be estimated.
1.5.2 Imposing the Supply Constraints

Let \( \psi \in \Psi \) denote the parameter vector of our model. Given \( v_{t+s}(h) \) and \( G_{t+s}(v) \), the implied supply of housing in period \( t+s \) is given by:

\[
R_{t+s}(h) = G_{t+s}(v_{t+s}(h))
\]

\[
= \int_0^1 \frac{1}{\tau_t} \left( \ln((a_{t+s}(1-\Phi(h+\eta))\beta(\nu_{t+s})) - \omega_{t+s})/\tau_{t+s} \right. \\
\left. \times e^{-x} x^{1-m_{t+s}} \frac{1}{2} \Gamma(1+1/m_{t+s}) \right) dx
\]

When considering a model with constant, time invariant supply:

\[
R_t(h) = R_s(h) \quad \forall s, \forall h
\]

I am imposing that supply elasticity \( \zeta = 0 \).

Note that the implied housing supply in equation (32) is a function of the parameters of our model, i.e. \( R_t(h) = R_t(h | \psi) \).

1.5.3 An Extremum Estimator

I define a moments estimator which matches quantiles of the income and value distributions while imposing the parameter constraints in Propositions 2 and 3 and the restriction that \( R_t(h) = R_s(h) \). Let \( \tilde{F}_{t,j}^N \) denote the jth percentile of empirical income distribution at time \( t \) that is estimated based on a sample with size \( N \). Similarly, let \( \tilde{G}_{t,j}^N \) denote the jth percentile of empirical housing value distribution at time \( t \) that is estimated based on a sample with size \( N \). Moreover, let \( F_t(y_{t,j}; \psi) \) and \( G_t(y_{t,j}; \psi) \) denote the theoretical counterparts of quantiles predicted by our model. For some weight \( W \in [0,1] \) define the objective function:

\[
L_N(\psi) = (1 - W) (l_{1y}^N(\psi) + l_{2y}^N(\psi) + l_{1g}^N(\psi) + l_{2g}^N(\psi)) + W \ln(\psi)
\]

(34)

where,

\[
l_{1y}^N(\psi) = \sum_{j=1}^1 \frac{1}{2} \left[ (F_t(y_{t,j}; \psi) - F_t(y_{t,j-1}; \psi)) - (\tilde{F}_{t,j}^N - \tilde{F}_{t,j-1}^N) \right]^2
\]

(35)

\[
l_{2y}^N(\psi) = \sum_{j=1}^1 \frac{1}{2} \left[ G_t(y_{t,j}; \psi) - G_t(y_{t,j-1}; \psi)) - (\tilde{G}_{t,j}^N - \tilde{G}_{t,j-1}^N) \right]^2
\]

(36)

\[
l_{1g}^N(\psi) = \sum_{j=1}^1 \frac{1}{2} \left[ G_2(y_{2j}; \psi) - G_1(y_{1j}; \psi) \right]^2
\]

(37)

The \( l_{1y}^N(\psi) \) term introduces the time invariant supply assumption using equation 32 in the considered periods. Our constrained estimator is then defined as:

\[
\hat{\psi}^N = \text{argmax}_{\psi \in \Psi} \ L_N(\psi)
\]

(38)
I use a parametric bootstrap procedure to estimate the standard errors, i.e. I parametrically bootstrap values of $\tilde{F}^N_{t,j}$ and $\tilde{G}^N_{t,j}$ and then implement the estimator above using the bootstrap percentiles. This approach, bootstrapping from the two marginal distributions, is consistent with the structure of our model.

More generally, I can consider models with changing supply, i.e. $\bar{\zeta} \neq 0$. The only difference is that the type of constrained imposed in estimation will change. Define

$$l^N_N(\psi) = \sum_{j=1}^J (|G_2(v_j; \psi) - R_2((h_j; \psi)|)^2$$

(39)

Let $r_1(h)$ be the density of $R_1(h)$. Then period two housing market equilibrium is given by:

$$R_2(h; \psi) = \int_0^h r_1(x) \left( \frac{V_2(x)}{V_1(x)} \right)^\zeta \, dx$$

(40)

and note that $r_1(h)$ is derivative of $R_1(h) = G_1(v)$ since $v = h$ in period 1.
2

ESTIMATION OF THE PRICE-QUALITY FRONTIER: AN ANALYSIS OF THE HOUSING MARKET IN THE GREAT RECESSION CRISIS IN MIAMI (FL) IN 2007

2.1 THE HOUSING MARKETS OF MIAMI (FL)

To illustrate the usefulness of the methods developed in chapter 1, I focus on Miami (FL) between 1995 and 2007. I divide this period into two sub-periods: 1) the pre-bubble period from 1995 - 2002; 2) the bubble period from 2002 - 2007. Figure 4 shows the growth in quantile and income for the pre-bubble period.

![Figure 4](image.png)

Figure 1: Growth in quantiles of income and rent for Miami 1995-2002.

I see that changes in income track changes in rents fairly well. This is not the case for the bubble period. In contrast, figure 2 shows the same graph for the period between 2002 and 2007. I find that changes
in rental rates dramatically outpaced changes in income. These figures suggest that the co-movements observed in the pre-bubble period between income and rent are not present in the bubble period. Our model uses income as the fundamental demographic characteristic affecting the demand, and thus the price, of housing. This decoupling of income and rent in the bubble period suggests that there is more going on than income in the large increase of prices, and that our basic specification of the model will not explain this change correctly. One of the possible changes this suggests would lie in the user-cost factor. I free this user cost for the bubble period in the estimation of section 2.5 and contrast it with the case where I take it as fixed in section 2.4.

Figure 2: Percentage Growth in quantiles of income and rent for Miami 2002-2007.

To illustrate the usefulness of the methods developed in this paper, I focus on Miami (FL) between 1995 and 2007. I divide this period into two sub-periods: 1) the pre-bubble period from 1995 - 2002; 2) the bubble period from 2002 -2007. Below I plot the Case-Shiller index from Miami for the last decade. From January 2002 through December 2007, the CPI increased 19 percent. During the same time period, the Case-Shiller index rose from about 140 to 275. The real increase is about \( \frac{275}{(1.19 \times 140) - 1} = 65 \) percent.
Our data sets are based on the American Housing Survey, which is conducted by field representatives who obtain information from occupants of homes. Interviewing occurs from May 30 through September 8. There are a national and a metropolitan version. I use the latter. The unit of observation in the survey is the dwelling together with the household. The sample is selected from the decennial census.\footnote{Due to incomplete sampling lists (and nonresponse), the homes in the survey do not represent all homes in the country. Therefore, the raw numbers from the survey are raised proportionally so that the published numbers match independent estimates of the total number of homes. Housing unit under-coverage and household nonresponse is about 11 percent. Compared to the level derived from the adjusted Census 2000 counts, housing unit under-coverage is about 2.2 percent.}

The AHS conducts surveys each year, but the metropolitan areas surveyed change from year to year. There is no fixed interval of repetition for surveying a given metropolitan area. The number of metropolitan areas surveyed has changed over time, likely due to changes in the AHS budget.

The Miami Metropolitan Area is defined by the Census in 1995 and 2002 to consist of Broward and Miami-Dade counties. In 2007, Palm Beach county is added to the definition of the Miami Metropolitan Area. In order to keep a constant definition of the metropolitan area across periods, I use micro data to construct the aggregates for 2007, so that only data for Broward and Miami-Dade counties are used in every period. Also, all dollar values are in 2007 dollars in this and our subsequent application.

I use income quantiles for the corresponding metropolitan areas. As discussed in the previous section, I aggregate the housing data for rental and owner occupied housing units. Since the AHS does not report rent paid by households net of utilities, I use reported housing costs and calculate the fraction of rent paid for utilities for those households that do not have them included in their rent payment. I then use this fraction to deduce the net rent of households with included utilities in their rent payment. Finally, I use polynomial re-
gression to extrapolate both data to common quantile bounds and aggregate.

A brief perusal reveals the following. Rates of appreciation for the overall market across our sample periods are similar to the appreciation found in the Case-Shiller index.\(^2\) Housing values appreciated by approximately 15 percent between 1995 and 2002. Moreover, our data provide little evidence of differential appreciation of houses of low relative to high value between 1995 and 2002. For the period between 2002-2007, the results are quite different. Ordering owner-occupied housing units by value in 2002, I find large housing price changes – ranging from approximately 30 percent at the 20th percentile to more than 70 percent for the top quantiles. Thus the data suggest that there is a lot of heterogeneity in housing price appreciation during the “bubble period.”

In contrast, the data indicate only moderate changes in rents during both periods. This is broadly consistent with the fact that real incomes increased on average by approximately 5 percent during the time with higher income earners experiencing larger gains. Rents increased faster during the bubble period than the pre-bubble period for the majority of rental properties. Nevertheless, rental changes were small in comparison to changes in home values. There is no evidence that rental changes were out of line with changes in fundamentals such as income and population growth.

With this brief overview of the data as a backdrop, I turn to the results obtained by estimating our model for the Miami market.

### 2.3 Empirical Results with Perfectly Durable Housing and Constant User Cost

The most simpler version of the model is specified for 2 periods, perfect durable housing. \((\zeta = 0)\) and constant user cost. I use a user-cost of capital factor of 0.09 for this period which is broadly consistent with historical averages for 30 year mortgage rates, inflation rates, property and income tax rates, as well as depreciation rates. This is the specification with the strongest assumptions. The figures show that even in this restricted version the model can fit the data well.

#### 2.3.1 The “Pre-Bubble” Period 1995-2002

Table 1 reports the parameter estimates and estimated standard errors for our model when supply side constraints are imposed.\(^3\) Table 1

---

\(^2\) Due to incomplete sampling lists (and nonresponse), the homes in the survey do not represent all homes in the country. Therefore, the raw numbers from the survey are raised proportionally so that the published numbers match independent estimates of the total number of homes. Housing unit under-coverage and household nonresponse is about 11 percent. Compared to the level derived from the adjusted Census 2000 counts, housing unit under-coverage is about 2.2 percent.

\(^3\) Note that I can estimate models with and without imposing the supply side constraints.
2.3 Empirical Results with Perfectly Durable Housing and Constant User Cost reports the estimates for the parameters of the income and housing value distributions and for the preference parameters.

Table 1: Estimates 1995-2002 for Miami: fixed supply and constant user cost.

<table>
<thead>
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<th></th>
<th>µ</th>
<th>σ</th>
<th>τ</th>
<th>β</th>
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Not surprisingly, our model fits the data very well as illustrated by the next four plots.

The figures show a good fit for both cities, as well as reasonable numbers for elasticities and shares of income spent in housing. Also, figure ?? show equilibrium in the quality market and satisfaction of the assumption of perfect durability of housing. Estimation for Portland, Pittsburgh and Boston are also provided in the appendix showing similar results. The fact that I are able to obtain good fit of the model, even under the assumption of perfect durability, suggests that this does not impose a very strong restriction.
Figure 4: Rent and income distribution fit for Miami: Fixed supply and constant user cost model.
2.3 Empirical Results with Perfectly Durable Housing and Constant User Cost

Figure 5: Equilibrium and Elasticities for Miami: Fixed supply and constant user cost model.
Next, I consider the “bubble” period 2002 -2007. Again I need to merge the rental distributions with the imputed rent distribution for owner occupied housing. While there were some differences in interest rates and inflation expectation between 1995-2002 and 2002-2007, I use the same user cost factor of 0.089 for this specification. Recall figure 2 shows that income growth and capital gains diverged during that period. I find that changes in capital gains (changes in imputed rents) clearly outpaced changes in income.

I then add the 2007 (bubble period) data to our model and reestimate the parameters of our model. Table 2 reports the parameter estimates and estimated standard errors for the parameters of the income and housing value distributions and for the preference parameters for all three periods. Again I are fixing the Poterba factor are 0.089 for all three periods.

Table 2: Estimates 2002- 2007 for Miami: fixed supply and constant user cost.

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While I can fit the income distributions in 1995, 2002 and 2007 as well as housing rental distributions in 1995 and 2002, I obtain a very poor fit the distribution of housing values in 2007. This illustrated by the next figure.

I, therefore, conclude that our model cannot fit the data for al three periods if I hold the Poterba factor constant for all three time periods.
2.3 Empirical Results with Perfectly Durable Housing and Constant User Cost

2.3.3 Quantifying the Extent of the “Housing Bubble”

There are two possible approaches suggested by our model that I can use to quantify the extent of the bubble.

First, I perform a counterfactual analysis to decompose the changes in the rental value distribution in 2007 into two components: a) a part that is explained by the observed changes in income; and b) a remainder term. I treat the income distribution in 2007 as exogenous. I estimate our model using data for the income distributions from all three time periods and the house value distributions for the first two time periods. Hence, I estimate the parameters using the objective function exactly as above except that I do not use the value data distribution in 2007. This identifies all of the model parameters for all three periods. I find a very close fit to all three income distributions and to the 1997 and 2002 house value distributions. The 2007 bubble can then be measured by the difference between predicted and actual rental value distribution in 2007.

The second approach is the estimation of the user cost, which I will cover in section 2.4 and 2.5.

Figure 7 plots the difference between the observed and the predicted housing price by housing quality percentile.

Our analysis suggests that the distribution of imputed rents cannot be explained by fundamentals in Miami in 2007 if I hold the Poterba fixed throughout all three time periods. From January 2002 through December 2007, the CPI increased 19 percent. During the same time period, the Case-Shiller index rose from about 140 to 275. The real increase is about (275/(1.19*140) -1) = 65 percent. Our estimates are somewhat lower than this, but not dramatically so.
The second approaches suggested by our model is to free the user cost factor. I can conduct the following thought experiment and ask ourselves how much I need to adjust the Poterba Factor in 2007 in order to fit the data. Recall that a key feature of the bubble period is that households may have had unrealistic expectations about the magnitude of the expected housing price appreciations and / or the risk premium associated with home ownership. Hence, the assumption that the Poterba factor should be treated as constant throughout the observation period may not be valid.

To capture these divergent expectations, I can free up the Poterba factor for the period 2007 and treat it as a parameter to be estimated. The idea here is that changes in the Poterba factor between 2002 and 2007 capture the changes in housing appreciation expectations and attitudes towards risk that occurred during that time period. Searching over all reasonable values for the Poterba factor in 2007, I can trace out the objective function of our estimator as a function of the Poterba factor. Figure 8 below illustrates our results.

I find that our estimate for the Poterba factor during the bubble period is approximately 0.046 which compares to 0.090 which is the value I used for the non-bubble periods in 1995 and 2002.

Table 3 reports the estimates for the parameters of the income and housing value distributions in 2007 when set the user cost factor to be equal to 0.046.

The next two figures then plot the implied income and housing rent distributions for 2007. I find that our model can now fit the income and housing rent distributions in all three periods, if I allow households to have different price appreciation and risk expectations during the bubble period.
Table 3: Estimates 2007 for Miami: fixed supply and estimated user cost

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<td>( \sigma )</td>
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<td>(0.1684)</td>
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Objective as a function of Poterba factor.

(a)

Figure 8: Estimation objective function as a function of user cost.

2.5 EMPIRICAL RESULTS WITH CHANGING HOUSING SUPPLY AND ESTIMATED USER COST

2.5.1 First Stage: The Rent-to-Value Ratios

I estimate the rent-to-value functions using our non-parametric matching estimator for the three periods in our data set. Figure 11 plots the estimated functions for the three time periods.

I find that the rent-to-value ratio was ranging between 0.07 and 0.06 in 1995. The function changed little between 1995 and 2002 with user costs slightly decreasing at the low and high end of the quality distribution. In contrast, I see large changes in the rent-value function during the bubble period between 2002 and 2007. The range of the function is from 0.035 to 0.046.

Note that the average 30 year mortgage rate was 7.95 percent in 1995, 6.54 percent in 2002 and 6.34 percent in 2007. Hence, credit became cheaper during the time period. In addition, there is widespread evidence that credit also became more available during the the bubble period, especially for first time home owners at the lower end of the distribution. Figure 2 shows that the largest changes in the rent-value

---

Note that it is difficult to estimate the locus outside a range of 50 and 500 thousand dollars due to sparseness of data in the sample outside this range.
ratio are for lower quality houses which is consistent with the notion that demand for these assets may have increased more strongly due to changes in credit markets.

Another possible explanation for the change in the user-cost function is that investor expectations about future appreciation in rental rates and, thus, housing values changed during that time period. Our estimates indicate the average expectations of annual real rental appreciation must have been on the order of 2 percentage points. Our non-parametric matching approach does not allow us to distinguish between the hypothesis that changes in the rent-value ratio were driven by changes in credit market conditions or by changes in investor expectations.

2.5.2 The Second Stage: The Aggregate Rent Distributions

Based on our estimates of the rent-value ratios, I can convert housing values into imputed rents. Merging the observe rent distribution and the imputed rent distribution for owner occupied housing I obtain the aggregate rent distribution for each time period. Figure 12 plot the changes in rents and the changes in income for the two key periods that I analyze in this paper.
Rent and rent loci: $k$ such that $E(v_t|y)=k$ $E(V_t|y)$.

2.5.3 The Third Stage: Nonlinear Pricing in Rental and Asset Markets

I implement the third stage of our estimation algorithm. Table 1 reports the parameter estimates and estimated standard errors, which are computed using a standard bootstrap algorithm. The standard errors account for the sequential nature of our estimation procedure and are corrected for the fact that the rent-to-value functions are also estimated with error. I report results for a model with constant housing supply and a model with varying housing supply. Since the later model is more compelling, I focus our discussion on the second model specification.

First, consider the parameter estimates of the utility function. Our estimates imply income elasticities that range from 0.60 to 0.72. Similarly, the price elasticities range from -1.1 for low income households to -0.67 for high income households. The income share of housing is 35 percent for low income households and 20 percent for higher income housing. I conclude that our housing demand estimates are plausible and broadly consistent with previous empirical evidence.

Next consider the parameter estimates for the income and aggregate rent distribution. Figure 13 plots the estimated and the observed income and rent distributions for the three years 1995, 2002 and 2007. Overall, I find that our model fits the data very well. These findings
support our expectation that the generalized log-normal distributions employed in our model are sufficiently flexible to capture closely the key regularities exhibited by the data.

Our estimate of the annual supply elasticity is approximately 0.021 with a standard error of 0.001. Figure 14 shows the resulting changes in housing supply. I find that housing supply percentage growth ranges between 0.7% and 3.7% for the 7 years of pre-bubble period. For the bubble period, the supply changes range between 3% and 5% for the 5 year period, which correspond to annualized changes between 0.5% and 1%. Overall, the changes are at the lower end of the estimates of supply elasticities summarized in Glaeser [2004].

Next I consider the changes in rents predicted by our model. Figure 4 shows that rents were fairly stable during the pre-bubble period between 1995 and 2002. They increased by moderate amounts and decreased at the lower end of the quality distribution. Rents increased at a faster rate during the bubble period with 5 year increases ranging between 2 and 6 percent. Our model of rental prices accounts for changes in real income, housing supply and population growth that occurred during the period from 1995-2007. Our findings are consistent with research by Sommer et al. [2011] who also report that there was no “bubble” in rental rates for housing.

Finally, I compute the implied predicted annualized capital gains during the pre-bubble and bubble periods. The predicted capital gains combine our estimates of the rent-to-value functions with the pre-
dicted equilibrium hedonic rent functions. The results are illustrated in Figure 5. First consider the pre-bubble period. Our estimates predicts 7-year capital gains were approximately 10 percent for most houses. Houses in the upper two deciles of the quality distribution had larger gain of up to 20 percent. During the bubble period, our model predicts 5-year capital gains ranging between 30 and 60 percent. I see larger increases at the low and middle levels of quality than at the high end (with the exception of the highest percentiles). This pattern of gains is consistent with the notion that loosening of credit market constraints played a role in explaining the run-up in housing markets. I would expect that this change in lending policies would primarily affect lower and middle quality housing units. However, our findings are also consistent with the hypothesis that some investors did not have realistic expectations about future price levels.

Note that the average 30 year mortgage rate was 7.95 percent in 1995, 6.54 percent in 2002 and 6.34 percent in 2007. Hence changes in the interest rate cannot explain this reduction in the user-cost factor. Given that tax rates as well as maintenance and depreciation rates were did not change much either during the time period, I explain this change in the user-cost factor by changes in investor expectations about future appreciation in housing rentals. Our estimates indicate
the average expectations of annual real rental appreciation must have been on the order of 3 to 4 percentage points.

2.6 CONCLUSIONS

I have developed a new approach for estimating the price-quality frontier in markets for durable housing. Our method has a number of advantages. First, it does not require any a priori assumptions about the characteristics that determine house quality. Second, it is easily implementable using metropolitan-level data on the distribution of house values and the distribution of characteristics of households. Third, it provides a straightforward summary of the changes in prices across the house quality distribution. In particular, I do not need to collapse the change in the distribution of prices into one number, as, for example, with the Case-Shiller index. Fourth, it gives insights into the mechanism that generates those price changes. Finally, I can use it to measure the extend of housing bubbles in local real estate markets.

I apply our methods to study the housing markets of Miami. I find that our model of nonlinear pricing in housing markets is consistent with the data observed in Miami – before and during the bubble period – if I allow households to have price appreciation expectations during the bubble period that may not be inline with later realiza-
tions of asset price changes. I find that the user cost factor must have dropped by almost 50 percent from the pre-bubble level to account for the large run-up in housing values that I observed in Miami during the bubble period. The subsequent fall in housing prices indicates that expectations driving this fall in user cost were not realized.
Part II

HOUSING MODELS WITH HETEROGENEOUS HOUSEHOLD PREFERENCES FOR QUALITY
3 ESTIMATING HEDONIC EQUILIBRIUM FOR METROPOLITAN HOUSING MARKETS WITH MULTIPLE HOUSEHOLD TYPES.

3.1 INTRODUCTION

A central challenge in modeling housing markets is treating heterogeneous quality. I understand quality as a measurement of the stream of housing services provided by one housing unit to the household occupying it. In contrast with other goods, this definition of housing consumption implies a difficulty in separating observed rents and values from unobserved quality. The hedonic literature has approached this issue by estimating a mapping from a vector of characteristics, each having some well-defined cardinality, to the observed value.\footnote{This approach has been fundamental in estimating the demand for characteristics of differentiated products. In literature specific to housing, hedonic theory and empirical work has advanced important contributions to determine value of different housing characteristics. Ekeland et al. [2004], Bajari and Benkard [2005], Berry and Pakes [2007], Heckman et al. [2010] and P. et al. [2012] have established identification of various versions of hedonic models under weak functional form assumption.}

However, measuring housing characteristics is in practice challenging, costly and requires some ex-ante assumptions about the characteristics that are relevant for the consumption decision. Furthermore, I might face unobserved heterogeneity that is systematically related to these observable characteristics. These issues are the motivation behind treating housing quality as unobserved by the econometrician in the present model.

Additionally, heterogeneity in tastes plays an important role in the behavior of households in the housing market. Sorting on income is an important characteristic of equilibrium sorting models, a strand of literature to which this model belongs. When sorting on income is contrasted against the data in the housing market, it is not very closely satisfied. However, when households are divided by simple demographic characteristics I observe that it is significantly better satisfied in the data, suggesting an important heterogeneity in preferences that is linked to these demographic differences. The literature has estimated differences in preferences for public goods and other neighborhood characteristics by group and by agent, along with in-
come as important determinants of sorting (Epple [1987], Ekeland et al. [2004])\textsuperscript{2} In our case I are only interested in preference differences that can be connected to demographic characteristics as mentioned above. These heterogeneous preferences are an important driver of price movements and link population demographic composition with demand distribution. Furthermore, structural parameters that characterize differences in preferences across demographic groups play an important role in the evaluation of policies and their heterogeneous welfare implications as discussed in Kuminoff et al. [2010].

Similar to Rosen [1974], I consider a standard hedonic equilibrium model with heterogeneous housing units whose quality can be captured by a one dimensional (latent) index. Rosen’s work makes use of the insight that the equilibrium pricing function can be characterized by the solution to a nonlinear differential equation that arises from the household’s optimization. Epple et al. [2013] present a parametrization of this model that yields tractable solutions for the equilibrium pricing function with latent quality. However, when preferences heterogeneity is introduced, their solution is no longer applicable. This model resorts to an alternative formulation of the model that partitions the housing quality vector and uses numerical methods to obtain the equilibrium pricing function.\textsuperscript{3} While sharing some characteristics, the current approach presents alternative advantageous features over that one. First, the current model allows for flexibility of utility functional forms. Second, a new solution and estimation algorithm is developed for this generalized class of models that is based on discrete approximations of the distribution of housing quality. Third, no assumptions on the form of the distributions of exogenous or endogenous variables is required. Fourth, the inclusion of several types allows for additional sources of observed heterogeneity, which allows the model to be used for the evaluation of welfare effect of policies with differential impact across demographic groups. Finally, evaluating the model jointly for more than one metropolitan area allows us to calculate compensating variations for different household types across cities.

Identification of the model is established taking into account that rental prices are not observed for all units, depending on tenure status. Treating quality as latent implies an identification problem if observe data in one market only (metropolitan area or time period) because I have no evident scale for housing quality. I observe payments for housing services, but not price and quality separately. For every model with equilibrium pricing function \(v(h)\), I can find a transformation of the utility function so that the equilibrium is observationally equivalent to a model with a linear pricing function. This allows us to implement a normalization that uses rents in a baseline period as our

\textsuperscript{2} Others have incorporated the endogeneity of the determination of some characteristics of the consumption good (i.e. race composition in a neighborhood) (Epple and Sieg [1999], Bayer and Timmins [2005])

\textsuperscript{3} The treatment of latent quality is also related to recent work on nonlinear pricing in housing markets by Landvoigt et al. [2011] who study the impact of credit constraints on house prices in San Diego.
measure of quality. Since I do not impose any parametric assumption in the rents distribution, the same applies to the initial distribution of housing quality. With this normalization I can identify preferences from the income expansion paths. Then, I can use observations on more than one period or metropolitan area to identify nonlinearities in the pricing function while maintaining the assumption that preferences be time or metropolitan area invariant.

An additional identification issue addressed is that rental rates and housing values are partially latent, i.e. for a single unit, the econometrician either observes the rental price or the value. For each quality level, I non-parametrically identify the rent-to-value ratio by characterizing the set of households that are indifferent between owning and renting at each level of housing quality. To identify pricing functions for house values, I make the simplifying assumption to separate the rental from the investment-ownership decision. This allows us to clarify the connection between values and rents in equilibrium: namely, that purchase values are determined by a expected net present value of the discounted stream of rental incomes of a unit. I extend the standard asset pricing results of Poterba [1992] to the case of heterogenous quality units. In particular, I model housing stock as owned at each point in time by investors who trade real estate assets. I estimate the rent-to-value functions using non-parametric matching estimators and use them to aggregate owners and renters data in a single market. This process allows to calculate price indices across time that are consistent across the markets with renters and owners.

As mentioned before, in contrast with Eppele et al. [2013], I can no longer derive a closed form solution for the pricing function under preferences heterogeneity. However, by calculating income cutoffs that define indifference between two quality levels at different prices, I can derive household and aggregate demands for any quality. In practice, I partition quality into discrete bins and obtain the share of households that would optimally locate in each bin. Then, I calculate the implied excess demand and choose prices that clear the market. I estimate the model using an algorithm that calculates equilibria for each set of preference parameters obtained from an extremum estimator that matches quantiles of rent distributions while imposing necessary structural constraints.

The first application of our model uses k-means clustering to identify household types in 2 dimensions, age of householder and number of children, according to their behavior in the market. This data mining technique allows us to learn structure from the data as an input for our structural model. I find 3 clusters: Very young households with few or no children, middle age households with children, and older households with no children (living in the household). In second place, I present a full estimation of the model defining types as households with and without children. This division satisfies our idea

---

4 Our assumption that households with similar incomes consume similar qualities is empirically explored in Henderson and Ioannides [1989].

5 Estimation is based on boundary indifference conditions in the spirit of Eppele and Sieg [1999].
of types because the presence of children is often a decisive variable when choosing a housing unit.

I estimate the model using periodic American Housing Surveys for 1999 and 2003 for New York City. Then, I include New York City and Chicago in a joint estimation. This will allow comparisons between distribution of housing qualities and the associated prices across metropolitan areas. Analysis of the differences in preferences allows to conclude, for example, that the presence of children lowers the preference for housing quality relative to other consumptions, which is accompanied by an increase in income and price elasticities.

3.2 Model

Our model considers a complete metropolitan area as one housing market. I consider both the rented and owner occupied units in a single market. Renter occupied units serve to model the purchase of housing services. Quality is another way to denote the amount of housing services provided by a housing unit. The household makes its housing consumption decisions every period in a frictionless market, given some exogenous income. In equilibrium, the price of each quality or level of housing services $h$ at period $t$ is the rent of the corresponding housing unit $v_t(h)$, which I allow to be nonlinear. At the same time, housing units with quality $h$ are purchased in asset markets at equilibrium values $V_t(h)$. I am separating the consumption and investment decision as a simplification. Yet, I maintain a connection between the two, given mainly by the assumption that houses are assets whose equilibrium value is a discounted stream of equivalent rents. Thus, I obtain pricing functions for rents and values that are internally consistent. I discuss the treatment of both markets in the following sections.

3.2.1 Housing Service Markets

I characterize housing quality by a one-dimensional ordinal measure denoted by $h$. There is a continuum of households whose mass can change across periods. Let us assume that there are $I$ observed types, i.e. households with and without children. The fraction of each type in period $t$ is given by $s_{i,t}$. Households differ in income denoted by $y$ and parametrically in their preference parameters by type. Households have preferences defined over housing services $h$ and a composite good $b$. Let $U_{i,t}(h, b)$ be the utility of a household of type $i$ at time $t$. Because households are paying for the housing services they derive from the housing unit in a certain time period, it is convenient to think of these households as renters. I also assume that they can change the quality of housing they consume in every period with no

6 Table 9 in Appendix B.1.2 illustrates the obtained quality index for one application and its relationship with several available housing infrastructure and neighborhood characteristics.
transaction or moving costs. I assume households receive an exogenous income that they spend every period.

Let $F_{i,t}(y)$ be the metropolitan income distribution at time $t$ of households of type $i$. The static simplification oversees the dynamics brought by the modeling of borrowing and lending. On the upper side, this assumption allows us to estimate the model with our data. For each metropolitan area, I have observations across time for a sample of housing units. However, the sample of households may change. Data limitations make it inaccurate to follow households across periods even if they did not change unit.

I can define a mapping $v_t(h)$ that denotes the period $t$ rental price of a house that provides quality $h$. Housing quality is ordinal so I can only define it up to a monotonic transformation. Given this pricing function, each household of type $i$ chooses consumption of housing quality $h$ at each period $t$ in order to maximize utility subject to each period’s budget constraint:

$$\max_{h_t, b_t} U_{i,t}(h_t, b_t) \quad (41)$$

$$\text{s.t. } y_t = v_t(h_t) + b_t$$

The first-order condition for the optimal choice of housing consumption is:

$$\frac{U_{i,h}(h_t, y_t - v_t)}{U_{i,b}(h_t, y_t - v_t)} = v_t'(h_t) \quad (42)$$

which gives the household’s housing demand $h_{i,t}(y_t, v_t(h_t))$ for each type $i$. The pricing function $v_t(h)$ should satisfy (42) for all types $i$ simultaneously, which implies that I do not have a closed form solution for relevant functional forms of the utility function.

However, relying on sorting by type, I can find an equilibrium pricing function once I partition the quality vector and implement a discrete approach. Sorting is implied by the single-crossing assumption on preferences, which states that within a type households with higher incomes are willing to pay more for a higher quality of housing.

Assumption 8 The utility function satisfies the following condition for each type $i$:

$$\frac{\partial}{\partial y} \left[ \frac{U_{i,h}(h_t, y_t - v_t)}{U_{i,b}(h_t, y_t - v_t)} \right]_{U_{i,t}(h, y - v(h)) = \bar{U}} > 0 \quad (43)$$

The single-crossing condition implies that there is stratification by income of households within each type in equilibrium. This stratification implies that $F_{i,t}(y) = G_{i,t}(v)$ is satisfied for each type $i$, where $G_{i,t}(v)$ is the distribution of rents and $F_{i,t}(y)$ is the distribution of income. Hence exists a mapping between income and rent for each type of household defined by a monotonically increasing function $y_{i,t}(v)$, or income loci, that will characterize sorting in equilibrium.
Again, introduction of heterogeneous preferences stop us from obtaining closed forms for the income loci as in Epple et al. [2013] from simple parameter restrictions on assumed distributional forms of income and rent. As a consequence, imposing a distributional form for the endogenous rent is no longer useful and I abstain from it.

Instead, I approach the solution of the model by partitioning the quality variable $h$, indexing it by $j = 1, ..., J$.

I can choose any partition of the quality values, and that will determine the quality bins for which I can calculate the aggregate demand. I index the pricing function accordingly, i.e. $v_j = v(h_j)$, or $v_{t,j} = v(h_{t,j})$ when a time period is being specified. Sorting implies that there exists cut-off points $\hat{y}_{i,j,t}$ such that the demand by type $i$ for quality $h_j$ in period $t$ as a function of prices (rents) is

$$H_{i,j,t}(v_{1,t}, ..., v_{J,t}) = F_t(\hat{y}_{i,j,t}) - F_t(\hat{y}_{i,j-1,t})$$

(44)

where $\hat{y}_{i,j,t}$ is defined as the level of income such that household type $i$ is indifferent between consuming quality $h_j$ and quality $h_{j+1}$ at their corresponding prices in period $t$, i.e. the level that satisfies the following for each type $i$:

$$U_t(h_j, \hat{y}_{i,j,t} - v_{j,t}) = U_t(h_{j+1}, \hat{y}_{i,j,t} - v_{j+1,t})$$

(45)

Note that there is no need to discretize the income distribution, to compute the housing demand in this model.

Moreover, the market clearing conditions can be written as a system of $J$ nonlinear equations in each period $t$:

$$\sum_{i=1}^{I} s_{i,t} H_{i,j,t}(v_{1,t}, ..., v_{J,t}) = r_j \forall j$$

(46)

where $s_{i,t}$ is the fraction of households of type $i$ at time $t$, and $r_j$ is the fraction of units in each quality bin $h_{j,t}$

$$r_j = R_t(h_j) - R_t(h_{j-1})$$

(47)

and $R_t(h)$ is the house quality distribution (cdc). For the moment, take supply of housing as given and inelastic.

Define $H_{i,j,t}^d(v_t)$ as demand of housing quality, given some prices $v_t$, up to quality $h_j$, i.e. the cumulative distribution for $H_{i,j,t}(v_t)$ in (46). Then, I define an equilibrium in the rental market as:

**Definition 2** A hedonic housing market equilibrium at a period $t$ is an allocation of housing consumption for each household and price function $v_t(h)$ such that

a) Households behave optimally given the price function;

b) Housing markets clear, i.e. for each period $t$, I have that for each level of housing quality $h_j \forall j$:

$$\sum_{i} s_{i,t} H_{i,j,t}(v_t) = R_t(h_j)$$

(48)

---

7 As the grid gets finer an finer, the discretized equilibrium should converge to the equilibrium with a continuum of housing choices.

8 I will implement normalization to identify the quality whose distribution is $R_t(h)$ in section 3.3. Also I extend the model to consider dynamic housing supply in section 3.2.3.
An equilibrium exists under standard assumptions discussed in the hedonic literature.

3.2.1.1 Preference parameterizations

Any parameterization could be used for the utility function in our method as long as it satisfies regularity conditions. Additionally, as long as there is a lower number of parameters than the number of quality bins in the partition, I can identify the structural parameters that correspond to preferences in the parameterization from observed data. I illustrate the elements I need to solve the model using the following utility function:

\textbf{Assumption 9} Let the utility provided by housing quality \( h \) for household type \( t \) at each period \( t \) be:

\[
U = u_t(h) + \frac{1}{\alpha_t} \ln(y_t - v_t(h) - \kappa_t)
\]  

(49)

with \( u(h) = \ln(1 - \phi_t(h + \eta_t)^{y_t}) \), where \( \alpha_t > 0 \), \( y_t < 0 \), \( \phi_t > 0 \), and \( \eta_t > 0 \).

This form is very flexible and gives rise to very flexible forms for the price and income elasticity equations. Finally, it also provides a closed form solution for the income cut-offs in (44).

\textbf{Proposition 9} The income elasticity of demand for type \( t \) is given by:

\[
\frac{dh}{dv} = \left( \frac{A_t(h)B_t(h) - B_t(0)A_t(h)}{\phi_t(h)} + v'(h) \right)^{-1} \frac{\phi_t(h + \eta_t)^{y_t}}{h} + v(h) + \kappa_t
\]

with \( A = v'(h)(1 - \phi + i(h + \eta_t)^{y_t}) \) and \( B = (-\gamma_i \phi_t h + (h + \eta_t)^{y_t})^{-1} \)

\textbf{Proposition 10} The approximate \(^{10}\) price elasticity of demand for each type \( t \):

\[
\frac{dh}{dv} = \frac{S_t(h)^2}{(-\alpha_i \phi_t y_t (y - \kappa_t)) (-1) S_t(h)} \left( \frac{1}{h} \right) \frac{-\alpha_i \phi_t y_t (y - \kappa_t)}{-\alpha_i \phi_t y_t h + (h + \eta_t)((h + \eta_t)^{y_t} - \phi_i) (h + \eta_t)}
\]

where \( S(h) = -\alpha_i \phi_t y_t h + (h + \eta_t)^{1-y_t} - \phi_i (h + \eta_t) \)

This specification of household preferences yields plausible price and income elasticities as I will see in our estimations. Recall that under heterogeneous preferences I do not obtain a closed form solution of the pricing function. Therefore, calculations of expressions \( v'(h) \) and \( v''(h) \) must obtained numerically.

Finally, I solve for the income cut-offs in (44) that will serve as the bounds that allow us to infer the demand for each quality:

---

9 This utility function requires the following two assumptions be satisfied \( 1 - \phi_t(h + \eta_t)^{y_t} > 0 \) and \( y_t - v_t - \kappa_t > 0 \). In estimation, these structural constraints are imposed.

10 I are assuming linear pricing \( (v(h) = p \times h) \) only for this calculation. The price elasticity of demand can also be derived without the linear assumption using numerical methods to both approximate the obtained pricing function and to solve for price in terms of the other elements of the model.
Proposition 11

\[ \hat{y}_{i,j} = \frac{v_j - e^{(M_{i,j+1}-M_j)\alpha_i} v_{j+1} + \kappa_i}{1 - e^{(M_{i,j+1}-M_j)\alpha_i}} + \kappa_i \]  

(50)

where \( M_{i,j} = \ln(1 - \phi_i(h_j + \eta_i)^{\gamma_i}) \)

The implied demands in are obtained by substituting these income cut-offs in the income distribution as described in (44)

Recall that the discrete approach to solution and estimation is very useful because it does not impose restrictions on the form of the utility function. The only restrictions on parameter values are imposed by the chosen utility functional form. Alternatively, to illustrate, I could use:

Assumption 10 Let utility for household type \( \text{i} \) be given by:

\[ U_i(h_j, b) = (y_t - v_t(h)) (h - \psi_t)^{\beta_i} \]

where \( \beta_i > 0 \). \(^{11}\)

Proposition 12 Under assumption 10, the income elasticity of demand for each type \( \text{i} \) is then given by:

\[ \frac{dh}{dy} y(h) = \left( \frac{C_i(h)D_i(h) - C'_i(h)D'_i(h)}{D_i(h)^2} + \nu'(h) \right) \frac{1}{h} v'(h) \frac{(h-\psi_t)^{\beta_i}}{\beta_i(h-\psi_t)^{\beta_i-1}} + \nu(h) \]

with \( C_i = \nu'(h)(h-\psi_t)^{\beta_i} \) and \( D_i = \beta_i(h-\psi_t)^{\beta_i-1} \)

Proposition 13 Under assumption 10, the price elasticity of demand for each type \( \text{i} \) is given by:

\[ \frac{dh}{dy} h = \frac{T_i(h)^2}{(-y)(-1)T'_i(h)} \left( \frac{1}{h} + \frac{y}{\beta_i(h-\psi_t)^{\beta_i-1}} \right) \]

where \( T_i(h) = h + \frac{(h-\psi_t)^{\beta_i}}{\beta_i(h-\psi_t)^{\beta_i-1}} \)

Proposition 14 Under assumption 10, the income cut-offs in (44) are given by:

\[ \hat{y}_{i,j,t} = \frac{v_{j+1,t} (h_{j+1} - \psi_t)^{\beta_i} - v_{j,t} (h_j - \psi_t)^{\beta_i}}{(h_{j+1} - \psi_t)^{\beta_i} - (h_j - \psi_t)^{\beta_i}} \]

(52)

I will initially take assumption 9 for our calculations because it is a more flexible function and provides flexible elasticities. As our examples illustrate above, the required elements for this approach can be obtained from general functional forms. Different forms will imply different levels of flexibility in the fitting of the data. The choice of the functional form will be an empirical issue, but no form that satisfies regularity conditions is ruled out by the discrete calculation of equilibrium and its estimation.

\(^{11}\)This utility function requires the following two assumptions be satisfied: if \( \beta_i > 0 \) then \( \psi_t \leq h \); and \( y_t - v_t - \kappa_t > 0 \).
### 3.2.2 Asset Markets

Our treatment of asset markets follows the one described in Epple et al. [2013], with the difference that I are again considering quality partitioned in $J$ sections. I provide a short outline here in order to provide a self-contained account of the relevant ingredients of the approach, and refer to the aforementioned paper for proofs. This approach is a generalization of Poterba [1984] to the case of more than one (in this case $J$) quality levels. Investors can borrow capital in short term bond markets. The one-period interest rate is denoted by $i_t$. Investors (owners) are also responsible for paying property taxes to the city. The property tax rate is given by $\tau_t^p$. Finally, owners incur in additional costs due to appreciation and maintenance at a rate $\delta_t$. Investors can buy and sell housing units of each quality $h_j$ at the beginning of each period. Let $V_t(h_j)$ denotes the asset price of a house of quality $h_j$ at time $t$ (while $v_t(h_j)$ is still the rent paid for quality $h$). Two assumptions describe the asset market and its investors.

**Assumption 11** Investors are risk neutral.

**Assumption 12** Asset markets are competitive.

Expected profits, $E_t[\Pi_t(h_j)]$, of buying a house of quality $h_j$ and then selling it in the next period are given by the sum of initial investment, flow profits from rental income at time $t$, and the discounted expected value of selling the asset in the next period. Assumptions 11 and 12 imply these expected profits must be zero:

$$E_t[\Pi_t(h_j)] = E_t \left[ -V_t(h_j) + v_t(h_j) + \frac{V_{t+1}(h_j)(1 - \tau_{t+1}^p - \delta_{t+1})}{1 + i_t} \right] = 0 \quad \forall j$$

Equation (53) suggest a relationship between rents and values of a housing unit with quality $h_j$. In fact, successive forward iterations show that

**Proposition 15** The asset value of a house of quality $h_j$ is the expected discounted flow of its future rental income:

$$V_t(h_j) = v_t(h_j) \left[ 1 + E_t \sum_{l=1}^{\infty} \beta_{t+1}(h_j) \right] \quad \forall j$$

$$= \frac{v_t(h_j)}{\beta_t(h_j)} \quad \forall j$$

(54)

where the quality specific discount factor $\beta_{t+1}(h_j)$ at time $t+1$ depends on interest rates, property tax rates and depreciation rates:

$$\beta_{t+1}(h_j) = \prod_{k=1}^{t} \frac{(1 - \tau_{t+k}^p - \delta_{t+k}) (1 + \tau_{t+k}(h_j))}{(1 + i_{t+k-1})}$$

(55)
with \(1 + \pi_t(h_j) = \frac{v_{t+j}(h_j)}{v_{t+j-1}(h_j)}\) denoting housing inflation at date \(t\) for quality \(h_j\). The user cost at quality \(h_j\) is defined as
\[
\hat{u}_t(h_j) = \frac{1}{1 + \sum_{l=1}^{\infty} \hat{\beta}_{t+l}(h_j)}
\]
(56)

This establishes a useful relationship between rents and values for all quality levels which I will exploit to identify the partially latent rents and values and to interpret the estimated user cost. In equilibrium, user cost can be calculated for any quality level in the partition of \(h\).

3.2.3 Housing Supply

Although durability of the housing plays an important role in housing markets, this assumption can be too strong for some metropolitan areas in some periods. I incorporate growth in supply following Epple et al. [2013], adjusting the theory to account for our partition of quality.

Let \(q_t(h_j)\) be the number of housing units of quality \(h_j\) at date \(t\). I normalize the population (of households, not people) to 1 in the first period \(N_1 = 1\). For further periods \(t > 1\), \(N_t\) is an exogenous process. In each period, I must have of course the same number of households and housing units in equilibrium. I use a reduced form formulation to model the available supply of housing of a certain quality in a certain period as a function of the previous available stock of that quality, and the change in value of the specific quality between periods:

Assumption 13 The supply function for quality \(h_j\) takes the form:
\[
q_t(h_j) = q_{t-1}(h_j) \left( \frac{V_t(h_j)}{V_{t-1}(h_j)} \right)^{p_\zeta} \forall j
\]
(58)

Population takes the form
\[
N_t = \sum_l q_{hl} \quad \forall t > 1
\]

\(P\) is the number of periods (years) between the observations I am considering, \(\zeta\) is an annual supply elasticity that I assume initially

\[\hat{u}_t(h_j) = \frac{1}{1 + \sum_{l=1}^{\infty} \hat{\beta}_{t+l}(h_j)}\]

When \(\tau_p, \delta, \pi,\) and \(i\) are small, our result approximates the continuous time solution of Poterba [1984]: \(u(h) = (1 + \tau_p + \delta - \pi(h))\).

Wheaton [1982] and Henderson and Venables [2009] incorporate durability in their treatment of housing. Besides modeling issues, this feature of housing markets plays an important role in the asymmetric reaction of metropolitan areas to economic booms and busts (Glaeser and Gyourko [2005], Brueckner and Rosenthal [2005]).
does not change across time. This reduced, constant-elasticity parametric form, is not derived explicitly from optimization of the supplier behavior. Still, it has advantageous properties: it reflects that the building decision is taken by suppliers with an interest of selling and therefore should be sensitive to the market value of a unit in the selected quality, \( V_t(h_j) \), and not the implicit rent. That logic implies that supply of a certain quality should increase if the correspondent value goes up, and should decrease when the opposite happens (reflecting depreciation and reduced incentive to invest in maintenance of these units). Finally, in terms of estimation and identification, it is desirable to have a parsimonious function that only introduces one additional parameter, \( \zeta \).

Our focus is to understand heterogeneity in preferences on the demand side. I do not introduce any role of the aggregate demographic composition of households or division into types in the supply side. Still, other literature has shown that beliefs about households demographic characteristics affect supply decisions. Furthermore, Ihlanfeldt and Mayock [2009] also study how demographics can affect the prices suppliers charge households of different demographic characteristics in a less competitive setup. Incorporating this into our model of supply would be an interesting extension.

Finally, I take the quality distribution, \( R_1(h_j) \), in an initial condition, \( t = 1 \), as given. It is defined by the market clearing condition for housing quality:

\[
R_1(h_j) = G_1(v_1(h_j)) \quad \forall j
\]

(59)

The distribution of quality in subsequent periods can be obtained by normalizing the number of units by the population in each period:

\[
R_t(h_j) = \frac{1}{N_t} \sum_{l=1}^{j} q_t(h_l)
\]

(60)

\[
= \frac{1}{N_t} \sum_{l=1}^{j} q_{t-1}(h_l) \left( \frac{V_t(h_l)}{V_{t-1}(h_l)} \right)^{p\zeta} \quad \forall j
\]

(61)

In \( t > 1 \), market clearing in the housing market at date \( t \) requires the following to be satisfied:

\[
R_t(h_j) = G_t(v_t(h_j)) \quad \forall j
\]

(62)

3.3 Identification

3.3.1 Rent-to-value function

In this section I relax the assumption that rents are observed for all households. I are making this assumption implicitly when I maximize utility of housing consumption for all households. In the data, they are actually not observed for owner-occupied housing and need to be imputed. In order to identify the rent to value locus I make the following assumption
Assumption 14 Let the quality $h^*_i,t,y$ consumed in equilibrium by households of type $i$ in period $t$ and income $y$ be the same regardless of tenure, i.e. households are indifferent between renting from themselves (i.e. living in owner occupied housing) or from a third person.

This assumption reflects the fact that tenure is not found in the preferences. More generally speaking, this also assumes that there is no sorting on unobservables into home ownership. Within types, preference parameters are fixed and households consumption is only a function of their income, holding prices fixed. Hence, in the model, households of a type $i$ with income $y$ consume the same housing quality optimally, independently whether they live in a rental unit, for which observe, $v_{i,t}(y)$, or live in an owner-occupied unit, for which I observe $V_{i,t}(y)$. As a consequence, I have the following result:

**Proposition 16** There exists an equilibrium locus that characterizes the rent at any quality level $h$, $v_t(h)$, as a function of its asset value, $V_t(h)$. Moreover, this function is non-parametrically identified.

In order to identify it the rent-to-value function, I vary income and trace the equilibrium locus between rents and values for each income level.\(^{15}\)

I can use this rent-value function to impute rents for owner-occupied. This loci is intimately linked to the user cost defined in proposition (15). The estimated rent to value function is an estimation of the user cost, in which I allow the user cost to change at each quality level.

In the data, rents and values are not perfectly correlated with income as implied by the model. However, I can use the means $E[v_{t|y}]$ and $E[V_{t|y}]$ to estimate the two sorting loci, $v_{t}(y)$ and $V_{t}(y)$, and use these to define the $V_t(v)$ locus. As a consequence the rent-to-value function is non-parametrically identified. This approach can be extended to models in which demands depends on a vector of observed state variables.\(^{16}\)

3.3.2 Quality

Since housing quality is ordinal and latent, there is no well defined unit of measurement for it if I only observe data for one period. This implies that I can use a normalization in a baseline period $t = 1$. I use the values of houses in this baseline as our measure of quality. Then, I can use observations on more than one period or metropolitan area to identify nonlinearities in the pricing function. This is similar to the case considered in Epple et al. [2013]. The next result formalizes the insight behind the normalization. Recall that $h$ is the vector of housing qualities that I are partitioning in values $h_j \ \forall j$. Similarly, $v(h)$ is the pricing function for any quality in the partition such that $v_j = v(h_j)$.

\(^{15}\) For this calculation, I can pool all observations across types, or calculate different loci by type and weight them according to the share occupied by each type.

\(^{16}\) An alternative strategy to identify and estimate the rent-to-value function or user cost locus is discussed is to use uses observations on units that were both rented and sold within a short period, as developed in Bracke [2013].
**Proposition 17** For every model with equilibrium pricing function $v(h)$, there exists a monotonic transformation of $h$ denoted by $h^*$ such that the resulting equilibrium pricing function is linear in $h^*$, i.e. $v(h^*) = h^*$ when the utility function of all types is transformed accordingly.

Proposition 17 then implies that if I only observe data in one given housing market and one time period, I cannot identify the utility function $(u_i(h) \forall i)$ separately from $v_1(h)$. A corollary of Proposition 17 is then that I can normalize housing quality by setting $h_j = v_1(h_j)$ in our baseline period $t = 1$. This normalization also implies that I can treat the prices in period 1 as known. In practice, when I choose the partition of $h$, which implies a partition of $v_1(h)$, it is convenient to choose a partition that occupies a relevant part of the distribution of observed aggregate rents.

Table 9 in appendix B.1.2 illustrates the connection between the equilibrium quality index and some observed infrastructure unit and neighborhood characteristics available in our data.

3.3.3 Preference Parameters, Pricing Functions and Supply

Define a parameter vector consisting of vectors of the preference parameters of each type as $\Theta_1 = (\theta_i, ..., \theta_I)$.

**Assumption 15** The parameters of the utility function do not change across time.

Then, define prices for each quality in the partition for each period $t > 1$ as $\Theta_{2,t} = (v_{1,t}, ..., v_{J,t}) \forall t$. Considering $T$ periods, under assumption 20, the full vector of parameters is $\theta = (\Theta_1, \Theta_{2,1}, ..., \Theta_{2,T}, \zeta) \forall t$. Recall $\zeta$ is the elasticity of supply. In our empirical implementation and for the rest of this section I consider 2 periods.

Given a set of preference parameters I can compute the income cut-offs $\tilde{y}_{i,j,1}$ in (52). For clarity, the income cut-offs and demand will also be indexed by time in this section since I am considering multiple periods in our identification analysis. For period 1, the predicted fraction of households of type $i$ that consume for housing type $j$ at time 1 is then given by:

$$H_{i,j,1}(\theta) = F_{i,1}(\tilde{y}_{i,j,1}) - F_{i,1}(\tilde{y}_{i,j-1,1}) \quad (63)$$

Moreover, let $H_{i,j}^N$ denote the observed fraction of households that of type $i$ that consume housing type $j$ at time 1. Note that given our discretization and normalization, this is equal to the observed fraction of households of type $i$ that pay rents between $v_{j,1}$ and $v_{j+1,1}$.

I can then form a set of orthogonality conditions based on:

$$m_{i,j,1}(\theta) = H_{i,j,1}(\theta) - H_{i,j}^N \quad (64)$$

These orthogonality conditions only depend on the parameters of the utility function $\Theta_1$ for the different types. Hence I have the following identification result:

---

17 Prices for period 1 are given by our choice of quality partition given the normalization $h = v_1$ discussed in section 3.3.2
Proposition 18 Given the normalization of housing quality, the parameters of the utility function can be identified / estimated based on data of the first period alone.

To identify and estimate the price function in periods $t > 1$, I exploit the observed sorting of households in these periods. The predicted fraction of households of type $i$ that consume for housing type $j$ at time $t$ is then given by:

$$H_{ij,t}(\theta) = F_{it}(\tilde{y}_{ij,t}) - F_{it}(\tilde{y}_{ij,t-1})$$

(65)

Since I have already identified the parameters of the utility function, demand in (65) only depends on price vector $\Theta_{2,t}$. Moreover, given prices in period $t$, $\Theta_{2,t} = (v_{1,2}, \ldots, v_{J,2})$, I can interpret $H_{ij,t}(\theta)$ as the fraction of households of type $i$ that live in rental units that cost between $v_{j,2}$ and $v_{j+1,2}$.

Let $H_{ij2}(\theta)$ denote the observed fraction of households that live in rental units that cost between $v_{j,2}$ and $v_{j+1,2}$.

Note that this fraction depends on the unobserved price vector in the second period. Similar to period 1, I identify the pricing function from orthogonality conditions based on:

$$m_{ij,t}(\theta) = H_{ij,t}(\theta) - H_{ij,t}(\Theta_{2,t})$$

for $t > 1$

(66)

Proposition 19 The pricing function in the periods $t > 1$ can be estimated based on the observed sorting of households by type in periods $t > 1$.

Finally I have that supply of housing is identified.

Proposition 20 Supply is identified for period $t = 1$ from the equilibrium condition and the normalization of quality. Also, supply is identified in $t > 1$ from the supply function as defined in (58). The parameter $\zeta$ is identified from equilibrium conditions.

The supply of type $j$ denoted by $r_{j,1}$ in period 1 is identified since

$$r_{j,1}^N = \sum_i s_{i,1}^N H_{i,j,1}$$

(67)

$R_{j,1}^N$ is given by the accumulation of $r_{j,1}^N$.

Note that in a model with constant supply, I can identify the supply in all periods. In a model with time dependent supply, (67) gives only the initial condition in the baseline period. For $t > 1$, I have identified the prices (propositions 18 and 19), and the rent-to-value functions, I have also identified the values $V_t(h_j)$ for each quality level. Therefore, once I identify $\zeta$, I can identify $R_{j,t}^N$ from

$$R_t^N(h_j) = \sum_{l=1}^j \frac{1}{N_t} q_{l-1}^N(h_1) \left( \frac{V_t(h_l)}{V_{l-1}(h_1)} \right)^{P\zeta} \forall j$$

(68)

where $q_{l}^N$ are determined by (69). Finally, $\zeta$ is identified from using (69) in the equilibrium condition

$$r_{j,t}^N = \sum_i s_{i,t}^N H_{i,j,t}$$

(69)

$r_{j,t}^N$ is of douse obtained from taking differences in $R_{j,t}^N$ is given by the accumulation of $r_{j,1}^N$. 

18
I estimate the model in 3 steps that follow the identification steps outlined previously. First, I estimate the rent and value loci using a non-parametric matching estimator. Appendix B.1.1 discusses this step in more detail and provides an example. The loci is closely connected to the user cost describe in 3.2.2, and I allow it to change for each quality value in the partition $h_j$ in each period $t$. The loci obtained defines a user cost and captures factors that are important in the relationship between rental and asset markets (i.e. credit market conditions, tax structures and investor expectations. Second, this loci is used to impute rents for owner occupied units. This is used to aggregate all units in one market. Third, I estimate the structural parameters of the model by matching observed quantiles of the aggregate rent distribution, while imposing market equilibrium as defined in (46), and structural restrictions specific to the chosen preference parametrization. This step uses an extremum estimator defined as a constrained minimum distance estimator.

I can follow the following algorithm to estimate the structural parameters while imposing equilibrium in the model:

i Choose some vector $\Theta$ that satisfies structural restrictions (i.e. the conditions in footnote (9) when choosing the utility function in assumption 9)

ii Calculate the implied income cutoffs $\hat{y}_{i,j,t}$ such that

$$U_i(h_j, \hat{y}_{i,j,t} - v_{t,j}) = U_i(h_{j+1}, \hat{y}_{i,j} - v_{t,j+1}) \quad \forall j, i, t$$

iii Calculate the implied demands from

$$H_{i,j,t}^d(v_{t,j}^*, \ldots, v_{t,J}^*) = F_{i,t}(\hat{y}_{i,j,t}) - F_{i,t}(\hat{y}_{i,j,t-1}) \quad \forall j, i, t$$

iv Calculate numerically the equilibrium implied prices in

$$\sum_{i=1}^I s_{i,t} H_{i,j,t}^d(v_{1,t}^*, \ldots, v_{J,t}^*) = r_{j,t} \quad \forall j$$

v Calculate differences between the implied and observed rent distributions by type (which is equivalent to comparing implied $v_{j,t}(\theta)$ and observed $v_{j,t}^N$ prices) in the following objective function:

$$Q_N^i(\theta, v^*) = \sum_{t=1}^T \sum_{i=1}^I \sum_{j=1}^J m_{i,j,t}^N(\theta)$$

where

$$m_{i,j,t}^N(\theta) = H_{ij,t}(\theta) - H_{ij,t}^N(\theta_2)$$

and
• \( H_{ij2}^N(\theta_2) \) denotes percentiles of the rent distribution observed in the data.
• \( H_{ij2}(\theta) \) denotes the theoretical counterparts of the quantiles predicted by our model.
• Supply in (70) is determined by the normalization of quality as in section 3.3.2. Supply in \( t > 1 \) is determined by (58), and it depends on \( \zeta \), a parameter that is also in the parameter vector being estimated \( \theta \).

vi Update \( \Theta \) until \( Q_N^N(\theta, v^*) \) is minimized.

I use a standard bootstrap procedure to estimate the standard errors. Alternatively, in practice, I can also use a simultaneous estimator. I stack all \( I \times (J - 1) \) orthogonality conditions of each period:

\[
m_t^N(\theta) = \begin{pmatrix} m_{11t}^N(\theta) \\ \vdots \\ m_{IJ-1t}^N(\theta) \end{pmatrix}
\]

and then for all periods. For instance, if \( t = 2 \), \( m_t^N(\theta) = (m_1^N(\theta)', m_2^N(\theta)')' \). I can then estimate the parameters of our model using a minimum distance estimator:

\[
\hat{\theta}_N = \text{argmin} \ m_t^N(\theta)' W_N m_t^N(\theta)
\]

for some positive weighting matrix \( A_N \).

Finally, note that I can identify the housing supply function and construct additional orthogonality conditions based on the market clearing conditions.

### 3.5 Empirical Results

#### 3.5.1 Data

I take data from the American Housing Survey, the most comprehensive national housing survey in the United States. It is conducted in the field from May 30 through September 8. There is a national and a metropolitan version, and, in selected years, also an extended metropolitan component for some metropolitan areas in the national version. I use the metropolitan version. There are surveys conducted every year, but the metropolitan areas covered in the metropolitan version change in each year. There is no fixed interval over which the same metropolitan area is surveyed again. The unit of observation in the survey is the housing unit together with the household. The same housing unit is followed through time, but the sample of households may change. Database limitations make it inaccurate to determine if the same household resides in the unit in different surveyed years, and impossible to track households that moved.\(^{19}\)

\(^{19}\)The sample is selected from the decennial census. Periodically, the sample is updated by adding newly constructed housing units and units discovered through coverage
As discussed in the previous section, I need to aggregate the housing data for rental units and owner occupied housing to implement our estimator. I estimate rent-to-value functions for each period and calculate the corresponding implied rents for owner-occupied units and aggregate. I use focus on the data for New York City metropolitan area in (1999, 2003). I include estimations for Chicago and Philadelphia in the same periods in an appendix to test the robustness of our results. The New York Metropolitan Area is defined by the Census in 1999 and 2003 to consist of the following counties: Bronx, Kings, Nassau, New York, Putnam, Orange, Queens, Richmond, Rockland, Westchester, and Suffolk counties (see figure (??)).

![New York Metropolitan Area](image)

(a)

Figure 16: Counties in Metropolitan Area.

Table 5: Percentages of Households by Characteristics.

<table>
<thead>
<tr>
<th></th>
<th>1999</th>
<th>2003</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Children</td>
<td>0.595</td>
<td>0.603</td>
</tr>
<tr>
<td>Children</td>
<td>0.404</td>
<td>0.396</td>
</tr>
<tr>
<td>Owners</td>
<td>0.325</td>
<td>6.7345</td>
</tr>
<tr>
<td>Renters</td>
<td>0.674</td>
<td>6.6668</td>
</tr>
<tr>
<td>White</td>
<td>0.556</td>
<td>0.635</td>
</tr>
<tr>
<td>Non-white</td>
<td>0.443</td>
<td>0.364</td>
</tr>
</tbody>
</table>

Table 16 shows some simple descriptive statistics that will be important for our empirical application. Initially, I will define household types as defined by k-means clustering. In a second application, I define types by the presence of children. Both criteria divides the improvement. The survey data is weighted because there is incomplete sampling lists and non response. The weights are designed to match independent estimates of the total number of homes. Under-coverage and nonresponse rate is approximately 11 percent. Compared to the level derived from the adjusted Census 2000 counts, housing unit under-coverage is about 2.2 percent.
population in large groups. The size of these groups mean that any significant differences I find between the behavior of the 2 groups will not have only a marginal effects in the market.

I are aware of the incidence that the recent run up in housing values and subsequent bust (largely interpreted as a bubble by the literature) could have on an analysis of housing markets that aims to connect fundamental variables like demographics and income to distributions of house rent and values. I use periods in which I believe the largest abnormal increases in prices had not happened yet with full force, as exemplified by the Case-Shiller index for NYC, and by the AHS, which closely follows the former when collapsed into a single index. Yet, this model could deal with extreme changes in values stemming from larger than normal price expectations or cheap credit, which would be detected specifically through movements in the user cost or rent-to-value functions as described in section 3.2.2. Epple, Quintero, and Sieg [2013] address directly the issue of the run up in values in years after the period I consider here and how it was reflected in user costs of households.

3.5.2 Household types: 3 types from k-means clustering.

k-means clustering is a standard method in data mining, originally from signal processing. The method partitions the points in a multidimensional data matrix into k clusters. An iterative partitioning minimizes the sum, over all clusters, of the within-cluster sums of point-to-cluster-centroid distances. The results presented here are for squared Euclidean distances. Similar results were obtained with cosine distance and are available.

(a) 3 Clusters, Euclidean Distance. New York, 1999.

Neill [2006] provide a clear treatment of this topic. Nath [2007] shows an interesting application of this method to detection of crime hot spots.
Table 6: k-means clustering centroids.

<table>
<thead>
<tr>
<th>Cluster</th>
<th># Children</th>
<th>Age</th>
<th>Share of Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1999</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.35</td>
<td>27.33</td>
<td>0.268</td>
</tr>
<tr>
<td>2</td>
<td>1.77</td>
<td>46.54</td>
<td>0.466</td>
</tr>
<tr>
<td>3</td>
<td>0.45</td>
<td>67.43</td>
<td>0.2646</td>
</tr>
<tr>
<td>2003</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.45</td>
<td>28.15</td>
<td>0.268</td>
</tr>
<tr>
<td>2</td>
<td>1.99</td>
<td>48.11</td>
<td>0.466</td>
</tr>
<tr>
<td>3</td>
<td>0.30</td>
<td>65.12</td>
<td>0.2646</td>
</tr>
</tbody>
</table>

Table 6 show the estimated clusters (in color) and centroids (black dots). Figure (??) shows a sample clustering. Using Euclidian distance, I can think of the centroids as the typical household in each particular cluster. In our case, I are doing an exploratory analysis of the data that I will then confirm with our model. I are choosing the clusters to be 3, although measures of the optimal number of clusters support this choice. Overall, I are interested in capturing the interaction of 2 very important variables that I believe influence greatly the economics decisions of households: age and number of children. They are very important variables for policy (i.e. affect rent control for older households) as well. Households are being classified with respect to their behavior in the housing market, captured by the share of income they spend on rent. I find a rather intuitive classification of the algorithm into 3 groups: young households with few or no children, middle age with more than one child, and older households with no children. I expect the last cluster to be comprised of a combination of households whose children have left their household already and a minority of those that never had children, but I tell the two apart from our data.

Figure 17: # of Children for each household type obtained through k-means clustering.

Figure (17) shows that age is the main factor defining the clustering. However, there are also important differences in the number of children in each group, the second group showing significantly more
children in the household. In the figure, the size of the circles represent the number of households represented by each plotted point.

(a) Distribution of tenure by type of households. New York City 1999.
(b) Distribution of tenure by type of households. New York City 2003.

Figure 18: Percentage of owners and renters for each household type obtained through k-means clustering.

Overall, New York City shows very large share of households that are renting when compared to any other city. Across types, younger households (type 1) are renters more often than households in the other 2 types, but differences change gradually (figure (18)). Figure (19) shows that households in the 1st and 2nd type have overall higher incomes.

When I look at values (Figure (34)) I see that type 2 households have overall more expensive houses as expected. However, higher rents are being paid by households of type 1. This corresponds to common knowledge about younger compared to older households.

3.5.2.1 Aggregate rents

As mentioned in section 3.3.1, the model incorporates both owners and renters in a single market, separating the consumption from the investment decision. The mapping between rents and values provided by the user cost allows our pricing functions to be consistent across both markets. In particular, our optimization implies that households with equal incomes will consume equal quality of housing whether they rent or own. Given that our data is noisy, I estimate this equivalence by calculating the mean rents and values by income (see figure(35) in Appendix B.1.1). These means give us the user cost $u(h)$. Given the estimation of the user cost at each income level, I use the estimated demand functions to express it as a function of quality.

The obtained user cost for both periods is between 0.05 and 0.08 for 1999, and 0.04 and 0.08 for 2003, across qualities. Figure (19) shows the resulting aggregate rents that include owners and renter occupied housing units in the market.

3.5.2.2 Equilibrium

Table 9 shows the estimated structural parameters. Figure 39 in Appendix ?? show the model does a good job of fitting observed distributions, which is a central part or our estimation strategy. The inverse
of $\alpha$ is related the preference for non-housing consumption. Our estimates show that the preference for this consumption is increasing in the types. This could reflect that the presence of children (for type 2) and aging (for type 3) can increase the desired demand for other goods and services brought by needs particular to these demographics. Inversely, this parameter implies type 1 exhibits a stronger relative preference for housing quality.

Another important parameter that seems to be driving most of the heterogeneity in preferences is $\eta$. $\eta$ is related to the utility observed at very low qualities of housing. In this case, this direct utility from very low qualities is highest for type 2, then for type 1, and finally for type 3. Similarly, the lowest quality for which a household would consume any positive level of housing is lowest for type 2, followed by type 1 and type 3. This would suggest that households of type 2 are more inclined to participate in the market than others, which is probably driven by the fact that the presence of children and their middle ages make them less likely to join other households. In other words, this can reflect a more available outside option for households of type 3 and 1 in that order, and therefore a higher opportunity cost of consuming very low qualities for them. Consuming no housing does not necessary mean homelessness. An alternative is to join another household (i.e. stay with their extended family or attend a nursing home for older age people).

Our estimate for the annual supply elasticity is 0.071. Recall that the changes in supply stock of a certain quality over time depending on the increase in values and the estimated elasticity $\zeta$, through the
Table 7: Estimates 1999-2003 for New York City with types defined by k-means clustering.

<table>
<thead>
<tr>
<th>Type</th>
<th>α</th>
<th>φ</th>
<th>η</th>
<th>γ</th>
<th>ζ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>2.089</td>
<td>3.904</td>
<td>5.652</td>
<td>-0.553</td>
<td>(0.0121)</td>
</tr>
<tr>
<td></td>
<td>(0.3113)</td>
<td>(0.0231)</td>
<td>(0.0122)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type 2</td>
<td>1.544</td>
<td>7.433</td>
<td>10.836</td>
<td>-0.901</td>
<td>0.071</td>
</tr>
<tr>
<td></td>
<td>(0.0133)</td>
<td>(0.0122)</td>
<td>(0.0319)</td>
<td></td>
<td>(0.003)</td>
</tr>
<tr>
<td>Type 3</td>
<td>1.134</td>
<td>7.234</td>
<td>0.0127</td>
<td>-0.904</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0176)</td>
<td>(0.0099)</td>
<td>(0.0131)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

supply equation (58). The implied supply growth by our ζ estimate ranges between 5% and 14% for the 4 year period, which correspond to annualized changes between 1.2% and 3.3%, with a mean of 1.9%. Figure (38) shows the relationship between the pricing functions, the value functions, and its effect on quality growth across time. The same figure shows that over the 4 year period the model implies an increase of around 10% in the total number of housing units across all qualities, with the largest number of additional units being created in the qualities around the middle of the distribution. Overall, the changes are along the lines of the estimates of supply elasticities summarized in Glaeser [2004].

Figure 19 shows the estimated pricing function for both periods considered. The figures for income rent loci (figure 21) and share of income spent on housing (figure 22) provide further insight on the differential behaviors across types. For most part of the distribution of income, type 3 pay higher rents of each level income (recall, for example, from figure (19) for type 3, almost 80% of the distribution earns under $60,000). Similarly, the share of income spent on housing is larger for this group. These behaviors depend on a combination of different preferences and income distributions by type. On the other side, type 2 exhibits lower rent payments at each income level, which of course implies a lower share of income spent on housing quality. This supports the previous interpretation of the differences in the preferences for housing quality found in the α parameter.

Finally, elasticities show that the type 3’s demand for housing quality is less sensitive to changes in both price and income. This might reflect more permanent housing consumption decisions of older households, but it is difficult to identify this effect with a static setup. Types 1 and 2 show similar elasticities, with type 2 being more sensitive for most of the distribution (again, recall that, even for types 1 and 2, approximately 80% of the households earn less than $100,000).

3.5.3 Household types: Presence of Children

Alternatively, I estimate the model choosing types based on ad-hoc characteristics that I believe intuitively are important for the deci-
Figure 20: Estimated pricing function using types obtained through k-means clustering.

Figure 21: Income loci using types obtained through k-means clustering.

sion of housing consumption. I use the presence of children to divide households into 2 types. This application serves as a way to contrast with the 3 type model presented before in 2 ways: to check whether adding additional types and demographic characteristics provides any significant gains that offset the additional computational cost; to see if the k−means clustering provides additional information that cannot be obtained by simply dividing households in discrete exogenously chosen types. Practically all studies that explore demand for housing or its characteristics by demographic types include children and race as possible determinants of type. Haurin and Kamara [1992] and Haurin, Hendershott, and Kim [1994] study the particular importance children have in housing decisions for female-headed households and young households respectively. I assume the number of children is exogenous to the household when making the decision of how much housing quality to consume. Other demographic characteristics like race may satisfy more closely this assumption of exogeneity. However, the presence of children is a interesting appli-
cation of the model because of the close intuitive connection it has with preference over housing characteristics. Besides the obvious issue of space, the presence of children creates different needs for housing unit and neighborhood amenities, even for households of similar sizes. At the same time, the presence of children is expected to raise demand for certain non-housing consumption. The characterization of these preferences is an empirical question. Studies have found few robust patterns, among which I have the higher willingness to pay for better school districts and additional rooms that households with children exhibit when compared to those without (Bayer, Ferreira, and McMillan [2007]). The role of children housing preferences over overall quality, which is what our latent quality index attempts to capture, is however not that clear.

In our sample, households with children consume slightly more expensive housing. However, they are also richer overall (see figure (24)). This is a case in which our sorting model is useful to calculate demand, taking into account the different income distributions, and identify differences in preferences.

Table 8 shows the preference parameter estimates with their corresponding standard errors, while figure (25) shows the estimated pricing functions. Figures (48) show that the estimated equilibrium implies parameters that fit the observed distributions rather well.

Table 8: Estimates 1999-2003 for New York City.

<table>
<thead>
<tr>
<th>Type</th>
<th>$\alpha$</th>
<th>$\phi$</th>
<th>$\eta$</th>
<th>$\gamma$</th>
<th>$\kappa$</th>
<th>$\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Children</td>
<td>1.224</td>
<td>6.832</td>
<td>2.505</td>
<td>-0.846</td>
<td>-1.405</td>
<td>0.068</td>
</tr>
<tr>
<td></td>
<td>(0.021)</td>
<td>(0.153)</td>
<td>(0.014)</td>
<td>(0.015)</td>
<td>(0.032)</td>
<td></td>
</tr>
<tr>
<td>Children</td>
<td>1.543</td>
<td>7.039</td>
<td>3.643</td>
<td>-0.848</td>
<td>-0.646</td>
<td>(0.004)</td>
</tr>
<tr>
<td></td>
<td>(0.0131)</td>
<td>(0.1811)</td>
<td>(0.0177)</td>
<td>(0.0191)</td>
<td>(0.009)</td>
<td></td>
</tr>
</tbody>
</table>

2 parameter estimates ($\eta$, $\alpha$) show particularly interesting differences between types. $\eta$ is larger for the households with children. This, together with the interaction with the other parameters, implies
that the utility derived directly from housing quality consumption at very low qualities is much lower for households without children; similarly, the minimum quality that a household requires to enter the housing market is lower for households with children. As mentioned before, this can reflect a more available outside option for households without children and should not necessarily be considered as going into homelessness. The presence of children would make households more likely to consume housing independently even at low qualities.

Also, the inverse of $\alpha$ is a parameter that affects the preference for consumption of the composite good. Larger $\alpha$ for the households with children tells us that the presence of children raises the preference for non housing consumption relative to other households, given income. This supports the hypothesis that, with limited resources, the presence of children leads households to increase their preference for other goods foremost. This effect is complementary with the tendency to consume lower qualities (affected by $\eta$ as discussed above).

Additionally, the estimated pricing functions show little action in the rental market. This suggests that the rent market followed closely
mild movements in the income distribution as shown in figure (24) and (25). Recall that the model aggregates all housing units in the same market regardless of tenure, and finds equilibrium rents for all.

The different behavior shown by the estimated parameters is more clearly exemplified by the income rent loci and share of income spent on housing in figures (26) and (27). For any income level, households with no children consume more expensive housing qualities. Furthermore, they spend a larger share of their incomes. This supports a stronger preference for housing quality of households with no children.

Finally, estimated elasticities show that households with children are less sensitive to price changes in their demand of housing, as well as less sensitive in changes in their income.

3.6 CONCLUSIONS

I have developed a new approach for estimating hedonic price functions for rents and values in an equilibrium framework where households have heterogeneous demands defined by demographics characteristics and quality is treated as a latent index. I provided a solv-

21 More details about the pricing function and its mapping to values are covered in section B.1.5.

22 The more expensive housing consumption by households with children observed in the aggregate data would be in this case a consequence of much higher incomes for the households with children than of stronger preferences for housing of this type.
3.6 Conclusions

Pricing Functions.

Figure 25: Estimated pricing function using types defined by presence of children.

Income Rent loci.

(a)

Income Rent loci.

(b)

Figure 26: Income loci using types defined by presence of children.

tion and estimation method for this class of models that is based on discrete approximations of the distribution of housing quality. Our method has a number of desirable features. First, it avoids the impossibility of obtaining a closed form solution for the pricing function under preference heterogeneity, and allows for flexibility of utility in functional forms. The method is also easily implementable using metropolitan-level data on the distribution of house values and rents for each household type, as well as the distribution of household income. Third, treating quality as latent does not require any a priori assumptions about the characteristics that determine house quality. Additionally, this allows us to provide a straightforward summary of the changes in prices across the house quality distribution. In particular, I do not need to collapse the change in the distribution of prices into one number, as, for example, the Case-Shiller index. I also deal with the issue of rents and values being partially latent. I estimate rent to value functions non-parametrically, and impute rents for owner occupied households to aggregate all households in metropolitan area
Fourth, no assumptions on the form of the distributions of exogenous or endogenous variables is required. Finally, the inclusion of several types allows for sources of observed heterogeneity. The model characterizes the interaction of different household types in a sorting model which maps demographic composition to the changes of equilibrium hedonic pricing functions. In particular, it provides estimations of demand and elasticities for each household type. This gives insights about the type specific preferences and makes the model suitable for welfare analysis of policies that affect households types differently.

I introduce an application in which I use $k$-means to cluster households into 3 types, which can be described roughly as older households with no children, younger households with no children, and middle aged households with children. I find a stronger preference for housing quality in the group of younger households, and interesting contrasting elasticities across groups. I also find that middle age households with children are more willing to enter the housing market even at very low qualities.

A second application divides households into types by the presence of children. The results in this case are surprising in terms of intuition of how households with and without children behave. Our results are robust across several metropolitan areas considered. The stronger preferences for housing found for households with no children may explain why other research has found lower effects of the presence of children than expected, i.e. children raise demand for school quality but that it does not increase demand significantly for other quality characteristics. I find that the presence of children actually raises more strongly the the desired levels of other forms of consumption, as suggested by Bayer, Ferreira, and McMillan [2007].
Price elasticities as a function of income.
NewYork.

Income elasticities as a function of income.

Figure 28: Elasticities using types defined by the presence of children.
Part III

HOUSING MODELS WITH MULTIPLE METROPOLITAN AREAS
4

COMPARING HOUSING QUALITY AND PRICE ACROSS MULTIPLE METROPOLITAN AREAS: A STRUCTURAL HEDONIC EQUILIBRIUM APPROACH.

4.1 INTRODUCTION

I have discussed how a central challenge in estimating models with heterogeneous housing is separating quality from price. One of the most advantageous features of the model in chapter 1 is its capacity to separate quality from price by tracing the changes in prices for different levels of the quality distribution through time. One natural generalization is the incorporation of multiple metropolitan areas. I exploit similar assumptions to those in the single metropolitan area model to obtain identification of the preference parameters and the pricing functions. In particular, I extend the time invariance assumption for preferences and assume preferences are invariant across metropolitan areas as well. This is useful to obtain identification in the multiple metropolitan area context. Identification relies on cross-sections of data for multiple metropolitan areas. Hence, I obtain identification by adopting a normalization in one metropolitan area and time period.

In this chapter, as in chapter 1, I characterize the price of housing as a function of quality for the entire housing distribution while treating quality as unobservable. The distribution of quality is fixed at a given metropolitan area. This distribution, coupled with the distribution of demand, gives rise to equilibrium price functions, called a hedonic price functions, determining price as a function of quality for the entire distribution of houses at each metropolitan area. The framework also characterizes the difference in the stock of quality between metropolitan areas as a function of the change in the equilibrium asset value of each quality type between metropolitan areas.

I estimate the model, using a 3-step estimator, for New York and Chicago simultaneously.

This chapter has some material developed for the paper A New Approach to Estimating Hedonic Pricing Functions for Metropolitan Housing Markets coauthored with Dennis Epple and Holger Sieg. I would to thank Martin Gaynor, Karam Kang, Yaroslav Kryukov, and seminar participants at CREST-ENSAE, CAE, ITAM, Baruch College/CUNY, and Carnegie Mellon for comments and discussions.

Identification proofs for the homogeneous type case are similar to those already presented in chapter 1 and are available upon request from the author.
4.2 Housing Markets Across Metropolitan Areas

The model discussed in Epple et al. [2013] can be changed to incorporate different metropolitan areas in a single period, and using the observations in different cities as a source of identification. The model for two different cities at a given point in time is analogous to the model for the same city at two different points of time. I will focus in this section on showing the basics of such a model for multiple metropolitan areas based on the one described in chapter 1. Details that go through unchanged are just mentioned briefly in the paper. The same goes for the proofs, which are all available upon request.

4.2.1 Asset Markets

I adopt a similar formulation of asset markets, with an asset market for each metropolitan area. Recall I make the following assumptions.

Assumption 16 Private investors are risk neutral.

Let \( V_{m,t}(h) \) denote the asset price of a house of quality \( h \) at time \( t \) and metropolitan area \( m \).

Assumption 17 Asset markets are competitive.

As before, the expected profits, \( \Pi_t \), of buying a house with quality \( h \) at the beginning of period \( t \) and selling it at the beginning of the next period are 0, since investors are risk neutral and entry into the profession is free, which implies a no-arbitrage condition. Solving for \( V_{m,t}(h) \), I obtain the following recursive representation of the asset value:

\[
V_{m,t}(h) = v_{m,t}(h) + \frac{(1 - \tau_{m,t+1}^p - \delta_{m,t+1})}{(1 + i_{m,t})} E_t [ V_{m,t+1}(h) ]
\]

where property taxes, \( \tau_{m,t+1}^p \), depreciation \( \delta_{m,t+1} \), and interest rates \( i_{m,t} \) are allowed to change across periods and metropolitan area.

Proposition 21 The user cost of capital

\[
V_{m,t}(h) = \frac{v_{m,t}(h)}{c_{m,t}(h)}
\]

is

\[
c_{m,t}(h) = \frac{1}{1 + E_t \sum_{j=1}^{\infty} \beta_{m,t+j}(h)}
\]

where

\[
\beta_{m,t+j}(h) = \prod_{k=1}^{j} \frac{(1 - \tau_{m,t+k}^p - \delta_{m,t+k}) (1 + \pi_{m,t+k}(h))}{(1 + i_{m,t+k-1})}
\]

2 Note that I can weaken this assumption. I only need to require that the marginal investor is risk neutral in each market.
The proof of this proposition follows the proof of 1 (see Appendix 7).

As before, our model does not necessarily assume that investors have correct expectations about housing rental appreciation. It is possible that expectations of rental price increases prove to be greater than the actual rates of increase that are realized.

4.2.2 Rental Markets

I develop a hedonic model of non-linear pricing in a rental market for housing services in which housing quality can be characterized by a one-dimensional ordinal measure denoted by \( h \). There is a continuum of households, with mass equal to one, which differ only in income \( y \) and whose preferences are unchanged across metropolitan areas. Let \( F_m(y) \) be the metropolitan income distribution at metropolitan area \( m \). Households have preferences defined over housing services \( h \) and a composite good \( b \).

Since housing quality is ordinal, housing quality is only defined up to a monotonic transformation. Given such a normalization, I can define a mapping \( v_m(h) \) that denotes rental price of a house that provides quality \( h \) in metropolitan area \( m \). All households are renters, and transactions cost in the rental market are zero. I do not model household movement across metropolitan areas, but I use the intuition that households can move between different metropolitan areas to make qualities and rent functions comparable across cities. Households are exogenously placed in each metropolitan area at the moment of estimation. It follows that a household in metropolitan area \( m \) makes its choice of housing to rent maximizing its utility:

\[
\max_{h_m, b_m} U(h_m, b) \quad \text{s.t. } y_m = v_m(h_m) + b
\]

where \( b \) denotes expenditures on a composite good. The first-order condition for each metropolitan area for the optimal choice of housing consumption is:

\[
s_m(h_m, y_m - v_m) = \frac{U_{h_m}(h_m, y_m - v_m)}{U_b(h_m, y - v_m)} = v'_m(h_m)
\]

Solving this expression yields the household’s housing demand \( h_m(y_m, v_m(h)) \) in metropolitan area \( m \). Integrating over the income distribution yields the aggregate housing demand \( H^d_m(h_m|v_m(h_m)) \) (the fraction of households whose housing demand is less than or equal to \( h_m \)).

To characterize household sorting in equilibrium, I impose, as before, that preferences satisfy a single crossing condition, which simply says that high-income households are willing to pay more for a higher quality house than low-income households. This assumption implies

**Proposition 22** If \( F_m(y) \) is strictly monotonic, then there exists a monotonically increasing function \( y_m(v) \) which is defined as

\[
y_m(v) = F^{-1}_m(G_m(v))
\]
y_m(\nu) fully characterizes household sorting in equilibrium.

The following functional form assumption delivers a closed form solution for the equilibrium pricing function

**Assumption 18** Income and housing are distributed generalized log-normal with location parameter (GLN) \(^4\).

**Proposition 23** If \(r_m = m_m^4\ \forall m\), the income housing value locus is given by the following expression:

\[
y_m = A_t (v_m + \theta_m)^{b_m} - \beta_t
\]

with \(a_m = \mu_m - \frac{\sigma_m}{\tau_m} \omega_m\), \(A_m = e^{a_m}\), and \(b_m = \frac{\sigma_m}{\tau_m}\).

Note that all of parameters of the sorting locus can be estimated directly from the data distributions.

I assume a similar functional form for household preferences as before, and I obtain a closed form solution for the pricing function. I obtain plausible price and income elasticities when estimating the model with these preferences.

Given this parametric specification of the utility function, I have the following equilibrium pricing function for each metropolitan area \(m\):

**Proposition 24** If \(b_m > 1\ (\sigma_m > \tau_m)\) and \(\kappa = \theta_m - \beta_m\ \forall m\), the hedonic pricing function is well defined and given by:

\[
v_m(h) = A_m \left[1 - (1 - \phi(h + \eta)\nu)^{\alpha(b_m-1)}\right]^{-\frac{1}{b_m}} - \theta_t
\]

for all \(h > (\frac{1}{\phi})^{\frac{1}{\gamma}} - \eta\)

### 4.2.3 Housing Supply

The analysis of supply differs conceptually from that in chapter 1 because normalization is only made in 1 metropolitan area and 1 period. The rest of the supply distributions are determined in reference to this one, both in the reference metropolitan area in other periods, or in other metropolitan areas in any periods.

The supply of housing is determined by and can be characterized by a distribution of house quality \(R_m(h)\). Let \(N_m\) denote the population of metropolitan area \(m\). I normalize the population in some reference metropolitan area \((m = 1)\) to be one: \(N_1 = 1\). Let \(q_m(h)\) denote the density of housing of quality \(h\) in for non reference \((m \neq 1)\) metropolitan areas \(m\).

**Assumption 19** For the reference metropolitan area \(m = 1\), I normalize quality \(v(h) = h\), implying \(R_{1,1}(h) = G_1(h)\).

For other metropolitan areas in the 1st period, I assume the supply function in the 1st period is determined by the implied rent distribution: \(R_{1,m}(h) = \)

---

3 See Appendix A.2

4 I impose these restrictions when estimating our model.

5 I impose these restrictions when estimating the model.
I adopt the following constant-elasticity parametric form for this supply function across periods $t$ for a metropolitan area $m$:

$$q_{m,t}(h) = k_{m,t} q_1(h) \left( \frac{V_{m,t}(h)}{V_{m,t-1}(h)} \right)^\zeta \tag{83}$$

where

$$k_{m,t} = \int_0^\infty q_1(h) \left( \frac{V_{m,t}(h)}{V_{m,t-1}(h)} \right)^\zeta dh \tag{84}$$

### 4.2.4 Equilibrium

I can define an equilibrium in the rental market for each metropolitan area as follows:

**Definition 3** A hedonic housing market equilibrium is an allocation of housing consumption for each household and a price function $v_m(h)$ such that

a) Households behave optimally given the price function;

b) Housing markets clear, i.e. for each level of housing quality $h$, I have:

$$H^d_m(h|v_m(h)) = R_m(h) \tag{85}$$

where $H^d_m$ is the aggregate demand for quality $h$ at metropolitan area $m$, and $R_m(h)$ is the supply of the same quality and metropolitan area.

An equilibrium exists under standard assumptions discussed in the hedonic literature. Our analysis of the rental markets provides an equilibrium characterization that determines the rental price of housing, $v_m(h)$, as a function of house quality, $h$. The market fundamentals determining $v_m(h)$ are the quality of the housing stock and the demand for housing services arising from the distribution of income in the metropolitan population.

### 4.2.5 Identification

Housing quality is ordinal and latent. Since there is no well-defined unit of measurement for housing quality I can use the values of houses in a baseline period as our measure of quality. Following proposition 4 in Epple et al. [2013], I can use arbitrary monotonic transformations of $h$ and redefine the utility function accordingly. I normalize housing quality by setting $h = v$ for the reference metropolitan area. If, in addition, I make the standard assumption that preferences do not changing across metropolitan areas, I can establish identification of the preference parameters

---

6 In contrast with the model in Epple et al. [2013], in which the normalization creates a market-specific quality measure, I here jointly estimate the model for multiple markets with a common normalization of housing quality. This permits to comparing house quality distributions across metropolitan areas.
**Assumption 20** The utility function is time invariant.

Assumption 20 implies

**Proposition 25** The parameters of our utility function and the price function in all metropolitan areas $1 + m, m > 1$ are identified.

Additionally, I have that

**Proposition 26** The parameters of housing supply function are identified if I observe the equilibrium for, at least, two metropolitan areas.

### 4.2.6 Aggregating renters and owners

Rents are not observed for owner-occupied housing and need to be imputed. I assume households are indifferent between renting from themselves (i.e. living in owner occupied housing) or from a third person and that consumption is only a function of income given prices. Also, households with income $y$ consume the same number of housing independently of whether they rent, for which observe, $v_m(y)$, or own, for which I observe $V_m(y)$. Changing income I can trace out the equilibrium locus $V_m(y)$.

**Proposition 27** There exists an equilibrium locus $v_m = v_m(V_m)$ which characterizes the rent of any housing unit as a function of its asset price. Moreover, this function is non-parametrically identified.

Our data is more noisy since rents and values are not perfectly correlated with income as predicted by our model. I can use $E[v_m|y]$ and $E[V_m|y]$ to estimate the two sorting loci, $v_m(y)$ and $V_m(y)$, which implies that the rent-to-value function is non-parametrically identified. The key assumption is that the average quantity of housing consumed conditional on observed characteristics is the same for owners and renters, i.e. no sorting on unobservables into home ownership. This assumption will be relaxed somewhat in the next section, where I consider a model with heterogeneity in preferences across household types. The sorting in that case will still based on observables but includes other demographic characteristics.

### 4.2.7 Asset Markets and User Cost

Asset markets and user cost are treated as in Epple et al. [2013]. Although our current has only one time period, I use the fact that decisions are made over time to define concepts like user cost and apply them to such a model. The following considerations apply to our one period model by setting $t = 1$. Let $V_t(h)$ denote the asset price of a house of quality $h$ at time $t$.

---

7 An alternative strategy to identify and estimate the rent-to-value function is discussed in Bracke [2013], who uses observations that were both rented and sold within a short period.

8 I thus treat a person that lives in an owner-occupied house as both a renter and an investor.
by i. Investors (owners) are also responsible for paying property taxes $\tau_m^p$ to the city. Finally owners have additional costs due to appreciation and maintenance $\delta$.

**Assumption 21** Asset markets are competitive.

The expected profits, $\Pi_m, t$, of buying a house with quality $h$ at the beginning of period $t$ and selling it at the beginning of the next period in metropolitan area $m$ is given by:

$$E_t[\Pi_m, t(h)] = E_t \left[ -V_{m, t}(h) + v_{m, t}(h) + \frac{V_{m, t+1}(h)(1 - \tau_m^p - \delta_{t+1})}{1 + i_t} \right]$$

(86)

Zero expected profits imply:

$$0 = E_t \left[ -V_{m, t}(h) + v_{m, t}(h) + \frac{V_{m, t+1}(h)(1 - \tau_m^p - \delta_{t+1})}{1 + i_t} \right]$$

(87)

Solving for $V_{m, t}(h)$, I obtain the following recursive representation of the asset value at time $t$:

$$V_{m, t}(h) = v_{m, t}(h) + \frac{(1 - \tau_m^p - \delta_{t+1})}{(1 + i_t)} E_{m, t}[V_{m, t+1}(h)]$$

By successive forward substitution of the preceding, I obtain:

$$V_{m, t}(h) = v_t(h) + E_{m, t} \sum_{j=1}^{\infty} \beta_{m, t+j} V_{m, t+j}(h)$$

(88)

where

$$\beta_{m, t+j} = \prod_{k=1}^{j} \frac{(1 - \tau_{m, t+k} - \delta_{t+k})}{(1 + i_{t+k-1})}$$

(89)

Finally, let $1 + \pi_{m, t}(h) = \frac{v_{m, t+j}(h)}{v_{m, t+j-1}(h)}$ denote the rate of housing inflation at date $t$ and define $\tilde{\beta}_{m, t+j}$ as follows:

$$\tilde{\beta}_{m, t+j}(h) = \prod_{k=1}^{j} \frac{(1 - \tau_{m, t+k} - \delta_{t+k})(1 + \pi_{t+k}(h))}{(1 + i_{t+k-1})}$$

(90)

Then:

$$V_{m, t}(h) = \frac{v_{m, t}(h)}{u_{m, t}(h)}$$

(91)

where $u_{m, t}(h)$ is the user cost of capital:

$$u_{m, t}(h) = \frac{1}{1 + E_t \sum_{j=1}^{\infty} \tilde{\beta}_{m, t+j}(h)}$$

(92)

Consider the time-invariant case studied by Poterba [1992]:

$$E_t \prod_{k=1}^{j} \frac{(1 - \tau_{m, t+k} - \delta_{t+k})(1 + \pi_{t+k}(h))}{(1 + i_{t+k-1})} = \left[ \frac{(1 - \tau_m^p - \delta)(1 + \pi(h))}{1 + i} \right]^j$$

(93)
When $\tau^p_m$, $\delta$, $\pi$, and $i$ are small, the preceding closely approximates the continuous time solution of Poterba (1984): $u_m(h) = (i + \tau^p_m + \delta - \pi(h))$.

Our model does not necessarily assume that investors have correct expectations about housing rental appreciation and it is possible that expectations of rental price increases prove to be greater than the actual rates of increase that are realized.

4.2.8 Estimation

I propose a three step estimator for our model. First, I estimate the rent-to-value functions using a non-parametric matching estimator. I estimate the rent-value function for each metropolitan area. This approach thus allows us to capture changes mainly in investor expectations in a flexible non-parametric way. Second, I impute rents for owner-occupied housing and estimate the joint aggregate distribution of rents and income for each time period. Third, I estimate the structural parameters of the model using an extremum estimator which matches quantiles of the income and value distributions while imposing the parameter constraints in Propositions 23 and 24 and the housing market equilibrium restriction that $R_{m+j}(h) = G_{m+j}(v_{m+j}(h))$ for $j \geq 1$.

Let $\tilde{F}_{m,j}$ denote the $j$th percentile of empirical income distribution at metropolitan area $m$ that is estimated based on a sample with size $N$. Similarly, let $\tilde{G}_{m,j}$ denote the percentiles of the rent distribution. Moreover, let $F_m(y_{m,j}; \psi)$ and $G_m(v_{m,j}; \psi)$ denote the theoretical counterparts of the quantiles predicted by our model. Our extremum estimator is then defined as:

$$\hat{\psi}^N = \arg\max_{\psi \in \Psi} L^N(\psi)$$

subject to the structural constraints. The objective function is:

$$L^N(\psi) = (1 - W) (\tilde{I}^N_y(\psi) + \tilde{I}^N_r(\psi)) + W I_h(\psi)$$

for some weight $W \in [0, 1]$ and:

$$\tilde{I}^N_y(\psi) = \sum_{m=1}^M \sum_{j=1}^J ([F_m(y_{m,j}; \psi) - F_m(y_{m,j-1}; \psi)] - [\tilde{F}_{m,j} - \tilde{F}_{m,j-1}])^2$$

$$\tilde{I}^N_r(\psi) = \sum_{m=1}^M \sum_{j=1}^J [G_m(v_{m,j}; \psi) - G_m(v_{m,j-1}; \psi)] - [\tilde{G}_{m,j} - \tilde{G}_{m,j-1}]^2$$

$$I_h(\psi) = \sum_{m=2}^M \sum_{j=1}^J ([G_t(v_{t,h_j}; \psi) - R_t(h_j; \psi)]^2$$

Note $W$ is weight that is assigned to the market clearing conditions. I use a standard bootstrap procedure to estimate the standard errors.
This model is estimated for New York and Chicago and a comparison is made between distribution of housing qualities and the associated prices across metropolitan areas. The AHS provides data for concurrent years, 1999 and 2003, for these two major metropolitan areas. Furthermore, AHS definitions of each of these metropolitan areas is unchanged across these two periods. I use New York metropolitan area in 1999 as the base for our normalization. Estimation follows the same three-step procedure. I estimate rent-to-value functions for all the quality distribution for each metropolitan area. Then, I compute the corresponding aggregate rent distributions. Finally, I obtain the structural parameters using an estimator that includes orthogonality conditions for predicted and observed income and rent percentiles for each metropolitan area.

Figure 29: Equilibrium.

The fit I obtain is as good as the obtained for the case of Miami in Section ?? After a detailed listing of the parameter estimates. It is of interest to note, however, that the esti-

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9 The years used in this analysis largely precede the “bubble” period in US housing markets. Our parameter estimates for this multiple-market analysis are essentially unchanged if I treat user cost as constant within each metropolitan area at each date.

10 Preference parameters must simultaneously satisfy structural constraints for all metropolitan areas considered.
mated utility function parameters imply price and income elasticities similar to those obtained for the Metropolitan area. Income elasticities for New York decline slightly over the range of income from .7 to .6, and price elasticities increase slightly from -.7 to -.6 over the income range.

To illustrate the implications of the model, I begin with comparisons of housing prices and housing stocks by quality. I then turn to an analysis that provides insight for comparing agglomeration economies across the two metropolitan areas. The population of the New York metropolitan area is roughly twice as large as that of the Chicago metropolitan area. Hence, I would expect higher prices in New York at every quality level. The hedonic price functions for the two metropolitan areas, shown for 2003 in Figure 31 confirm this expectation. Points a, b, c, and d show house quality and annualized rent paid by households at the 20th, 40th, 60th, and 80th percentiles of the income distribution in Chicago at the optimally chosen housing consumption levels for those households. The corresponding uppercase values A, B, C, and D show qualities and prices that households with those incomes would optimally choose if they were located in New York. At each income level, households pay more in New York than Chicago, and consume lower quality housing in New York than Chicago. In considering these comparisons, it is important to keep in mind that our house quality measure is comprehensive and includes
Figure 31: Points a, b, c, and d show the quality consumption and rent paid in Chicago by households at the 20th, 40th, 60th, and 80th percentiles of the income distribution. Points A, B, C, and D represent the household’s behavior with the same incomes in New York.

I next turn to a comparison of the distributions of housing qualities in the two metropolitan areas. As I have just seen, at every income level, a household in New York consumes lower quality than the corresponding household in Chicago. The effect of this difference in consumption levels is to shift the distribution of quality in New York to the left relative to that in Chicago. This effect is augmented by differences in the income distributions in the two cities. The income distribution in Chicago first-order stochastically dominates the income distribution in New York for each of our two time periods. This relatively higher concentration of low-income households in New York accentuates the leftward shift in the quality distribution in New York relative to Chicago. The distributions of housing (numbers of housing units) by quality are shown in Figure 32. Chicago has relatively more high quality housing. Given its much higher population, however, New York has a larger number of housing units at almost at all quality levels than Chicago.

I find the annualized housing supply elasticity to be .055. This is a larger elasticity than I found for Miami, but recall that the elasticity for Miami was estimated over a period with dramatic run-up in housing prices. For smaller price changes, such as those for our sample period in New York and Chicago, the Miami supply elasticity would likely be higher.

I next turn to an investigation of agglomeration economies in New York relative to Chicago. In figure 33, I plot compensating variation that would be required for household of a given income in Chicago.
to be equally well off in New York. For a household at the 20th income percentile in Chicago, compensating variation of approximately 25% of the household’s income ($6,000) would be required. For a household at the 80th percentile, CV of approximately 16% of income ($15,000) would be required. These CV values can be thought of the minimum additional compensation in New York that would be needed to compensate households for the higher prices in New York relative to prices in Chicago. Put differently, productivity, and hence earnings, in New York would need to be higher by these amounts to compensate a household for the differences in housing price functions between the two metropolitan areas.
Figure 32: New York has a larger number of housing units, as expected from its larger population. When similar qualities are priced equally, the large differences in the value of the housing stock are reduced.
Compensating Variations as a percentage of income when moving from Chicago to New York, 2003.

(a)


(b)

Figure 33: Given higher prices in New York City, a household with income \( y \) will consume a higher quality \( h \) in Chicago. A positive compensating variation would be required to compensated households if moved from the Chicago to the New York housing market.
Part IV

APPENDIX
A.1 PROOFS

Proof 1 The single-crossing condition implies that there is stratification of households by income in equilibrium. Stratification implies that there exists a distribution function for house values \( G_t(v) \) such that:

\[
F_t(y) = G_t(v)
\] (95)

Hence there exists a monotonic mapping between income and housing value. If \( F_t \) is strictly monotonic, it can be inverted, and hence \( F_t^{-1} \) exists. Q.E.D.

Proof 2 Equating the quantiles for income and value distributions, i.e. setting \( F_t(y_t(v)) = G_t(v) \) for \( y_t > \exp(\mu_t) - \beta_t \) and \( v_t > \exp(\omega_t) - \theta_t \), yields:

\[
\int_0^{(\ln(y_t+\beta_t)-\mu_t)/\sigma_t} e^{-t \tau_t} \frac{1}{\tau_t - 1} dt = \int_0^{(\ln(v_t+\theta_t)-\omega_t)/\tau_t} e^{-t \tau_t} \frac{1}{\tau_t - 1} dt
\]

Assuming \( \tau_t = m_t \) in each period, the quantiles are equal when

\[
\frac{\ln(y_t+\beta_t)-\mu_t}{\sigma_t} = \frac{\ln(v_t+\theta_t)-\omega_t}{\tau_t}
\] (96)

Similar steps lead to the same conclusion when \( y_t < \exp(\mu_t) - \beta_t \) and \( v_t < \exp(\omega_t) - \theta_t \). Solving (96) yields:

\[
y_t = e^{(\mu_t-\frac{\sigma_t}{\tau_t} \omega_t)}(v_t+\theta_t)^{\frac{\sigma_t}{\tau_t}} - \beta_t
\] (97)

Q.E.D.

Proof 3 The household's FOC is:

\[
\alpha u'(h) \cdot dh = \frac{dv}{(y_t-v_t-\kappa)}
\] (98)

Substituting the income loci (81):

\[
\alpha u'(h) \cdot dh = \frac{dv}{A_t(v_t+\theta_t)^{b_t} - \beta_t - v_t - \kappa}
\]

Since \( \kappa = \theta_t - \beta_t \) \( \forall t \), the FOC becomes:

\[
\alpha_t u'(h) \cdot dh = \frac{dv}{A_t(v_t+\theta_t)^{b_t} - (v_t+\theta_t)}
\] (99)
Integrating the right hand side yields:

\[
\int \frac{dv}{A_t(v + \theta_t)^{b_t} - (v + \theta_t)} = \frac{1}{b_t - 1} \left( \ln \left( \frac{1}{A_t} \left( v_t + \theta_t - A_t(v + \theta_t)^{b_t} \right) \right) - b_t \ln v + \theta_t \right) + c_t
\]

which implies:

\[
\alpha u(h) = \frac{1}{b_t - 1} \ln \left( 1 - \frac{(v_t + \theta_t)^{1-b_t}}{A_t} \right) + c_t
\]

Notice that integrating the left hand side recovers the original function \( u(h) \).

Using the utility function we get

\[
\alpha \ln(1 - \phi(h + \eta)^\gamma) = \frac{1}{b_t - 1} \ln \left( 1 - \frac{(v_t + \theta_t)^{1-b_t}}{A_t} \right) + c_t
\]

Solving for \( v_t \)

\[
(1 - \phi(h + \eta)^\gamma)^{\alpha(b_t-1)} = \left( 1 - \frac{(v_t + \theta_t)^{1-b_t}}{A_t} \right) e^{c_t}
\]

and hence

\[
v_t = \left( A_t \left[ 1 - \frac{(1 - \phi(h + \eta)^\gamma)^{\alpha(b_t-1)}}{e^{c_t}} \right] \right)^{\frac{1}{1-b_t}} - \theta_t
\]

Normalizing the constant of integration to \( c = 0 \) gives the result. Q.E.D.

**Proof 4** We can write the household’s optimization problem as:

\[
\max_h u_1(h) + u_2(y - v(h))
\]

The FOC of this problem with respect to \( h \) is given by:

\[
u_1'(h) - u_2'(y - v(h)) v'(h) = 0
\]

Now define \( h^* = v(h) \) and hence \( h = v^{-1}(h^*) \). The decision problem associated with this model is then

\[
\max_{h^*} u_1(v^{-1}(h^*)) + u_2(y - h^*)
\]

and the FOC with respect to \( h^* \) is

\[
u_1'(v^{-1}(h^*)) v^{-1'}(h^*) - u_2'(y - h^*) = 0
\]

Now \( h = v^{-1}(h^*) = v^{-1}(v(h)) \) and hence \( v^{-1'}(h^*) v'(h) = 1 \). Hence we conclude that the two models are observationally equivalent. In the first case, we have non-linear pricing and in the second case we have linear pricing. Q.E.D.
Proof 5 Recall from our discussion following Proposition 2 that parameters $\Lambda_t$, $b_t$, $\theta_t$ can be estimated directly from data for income and house rent distributions. We show these are sufficient for identification of the utility function parameters. First consider the normalization $v_t(h) = h$. Recall that the equilibrium hedonic pricing function is given by:

$$v_t = \left( \Lambda_t \left[ 1 - [1 - \phi(h + \eta)^\gamma]^{\alpha(b_t - 1)} \right] \right)^{\frac{1}{1-b_t}} - \theta_t \quad (108)$$

Setting

$$\alpha = \frac{1}{b_t - 1} \quad (109)$$

implies

$$v_t = (\Lambda_t [1 - [1 - \phi(h + \eta)^\gamma]])^{\frac{1}{1-b_t}} - \theta_t = (\Lambda_t \phi(h + \eta)^\gamma)^{\frac{1}{1-b_t}} - \theta_t \quad (110)$$

Setting

$$\phi = \frac{1}{\Lambda_t} \quad (111)$$

implies

$$v_t = ((h + \eta)^\gamma)^{\frac{1}{1-b_t}} - \theta_t \quad (112)$$

Setting

$$\gamma = 1 - b_t \quad (113)$$

implies

$$v_t = (h + \eta) - \theta_t \quad (114)$$

Finally, setting

$$\eta = \theta_t \quad (115)$$

implies.

$$v_t = h \quad (116)$$

That establishes identification of the parameters of the utility function. The price equation in period $t+s$ is then given by:

$$v_{t+s}(h) = \left( \Lambda_{t+s} \left[ 1 - [1 - \phi(h + \eta)^\gamma]^{\alpha(b_{t+s} - 1)} \right] \right)^{\frac{1}{1-b_{t+s}}} - \theta_{t+s} \quad (117)$$

The parameters of joint value and income distribution in period $t$ nail down the parameters of the utility function. The assumption of constant utility then imply that $v_{t+s}(h)$ is fully identified by the parameters $b_{t+s}$, $\Lambda_{t+s}$, and $\theta_{t+s}$. Q.E.D.

Proof 6 The result follows from the discussion in the text.
**Proof 7** Recall

\[ V_t(h) = v_t(h) + E_t \sum_{j=1}^{\infty} \beta_{t+j} v_{t+j}(h) \]  

(118)

Substituting

\[ \bar{\beta}_{t+j}(h) = \prod_{i=1}^{j} S_{t+i}(1 + \pi_{t+i}) \]  

(119)

into the user cost

\[ u_t(h) = \frac{1}{1 + E_t \sum_{j=1}^{\infty} \bar{\beta}_{t+j}(h)} \]  

(120)

and expanding a few terms we get

\[ V_t(h) = v_t(h) \left( 1 + E_t \sum_{j=1}^{\infty} \bar{\beta}_{t+j}(h) \right) \]

\[ = v_t(h) \left( 1 + E_t \sum_{j=1}^{\infty} \prod_{i=1}^{j} S_{t+i}(1 + \pi_{t+i}) \right) \]

\[ = v_t(h) + v_t(h)(E_t[S_{t+1}(1 + \pi_{t+1}) + S_{t+1}S_{t+2}(1 + \pi_{t+1})(1 + \pi_{t+2}) + S_{t+1}S_{t+2}S_{t+3}(1 + \pi_{t+1})(1 + \pi_{t+2})(1 + \pi_{t+3}) + ...]) \]

\[ = v_t(h) + v_t(h)(E_t[S_{t+1} v_{t+1} S_{t+2} v_{t+2} + S_{t+1}S_{t+2}S_{t+3} v_{t+3} + ...]) \]

\[ = v_t(h) + (E_t[S_{t+1}v_{t+1} + S_{t+1}S_{t+2}v_{t+2} + S_{t+1}S_{t+2}S_{t+3}v_{t+3} + ...]) \]

which is equivalent to 118.

In the case of time-invariant user cost:

\[ E_t \prod_{i=1}^{j} \frac{(1 - \tau^p_{t+i} - \delta_{t+i})(1 + \pi_{t+i}(h))}{(1 + i_{t+i})} = \left( \frac{(1 - \tau^p - \delta)(1 + \pi(h))}{1 + i} \right)^j \]  

(121)

Then the user cost can be written as:

\[ u_t(h) = \frac{1}{1 + E_t \sum_{j=1}^{\infty} (S(1 + \pi))^j} \]  

(122)

With this time-invariant user cost, the rate of house value appreciation equals the rate of appreciation of rentals:

\[ \frac{V_{t+1}(h)}{V_t(h)} = \frac{v_{t+1}(h)}{v_t(h)} = (1 + \pi(h)) \]  

(123)
The infinite series can be now simplified to \( \sum_{j=0}^\infty (S(1 + \pi))^j = S(1 + \pi) \sum_{j=0}^\infty (S(1 + \pi))^j = S(1 + \pi) \frac{1}{1 - S(1 + \pi)} \). The user cost becomes

\[
\begin{align*}
    u_t(h) &= \frac{1}{1 + E_t S(1 + \pi) - S(1 + \pi)} \\
    &= 1 - E_t S(1 + \pi) \\
    &= 1 - E_t \left[ \frac{(1 - \tau^p - \delta)}{1 + i} \right] (1 + \pi) \\
    &= 1 - E_t \left[ \frac{(1 - \tau^p - \delta) + (\pi - \pi\tau^p - \pi\delta)}{1 + i} \right]
\end{align*}
\]

(124)

When \( \tau^p, \delta, \pi, \) and \( i \) are small we can approximate \( \pi\tau^p \approx 0, \pi\delta \approx 0 \)

\[
\begin{align*}
    u_t(h) &\approx E_t \left[ \frac{1 + i - (1 - \tau^p - \delta) - (\pi)}{1 + i} \right] \\
    &\approx E_t \left[ \frac{i + \tau^p + \delta - \pi}{1 + i} \right]
\end{align*}
\]

(125)

The preceding closely approximates the continuous time solution of Poterba (1984): \( u(h) = (i + \tau^p + \delta - \pi(h)). \)

**Proof 8** Given our normalizations, we have also identified the housing supply function in the first period since \( R_1(h) = G_1(v) \) which then identifies the density of housing quality in the first period \( q_1(h) \).

Proposition 5 implies that \( v_2(h) \) is identified. As a consequence \( G_2(v_2(h)) \) is identified. Proposition 6 implies that \( V_1(h) \) and \( V_2(h) \) are identified. As a consequence \( \zeta \) is identified of the market clearing condition:

\[
R_2(h) = k_2 \int_0^h q_1(x) \left( \frac{V_2(x)}{V_1(x)} \right)^\zeta dx
\]

(126)

Q.E.D.

Note that this proof generalizes for more complicated parametric forms of the supply function.

**A.2 THE GENERALIZED LOGNORMAL DISTRIBUTION WITH LOCATION (GLN4)**

The generalized lognormal distribution with location GLN4 pdf is given by:

\[
f(y) = \frac{1}{2(x + \beta) \tau^2 \sigma \Gamma \left( \frac{1}{r} + \frac{1}{r} \right)} e^{-\frac{1}{2} \tau^2 \sigma^2 \left( \ln(y + \beta) - \mu \right)^2}
\]

(127)

The CDF of the GNL4 distribution is given by:

\[
F_t(y) = \begin{cases} 
\frac{\Gamma \left( \frac{1}{r}, B(y + \beta) \right)}{2 \Gamma \left( \frac{1}{r} \right)} & \text{for } y < \exp(\mu) - \beta, \\
\frac{1}{2} & \text{for } y = \exp(\mu) - \beta, \\
\frac{1}{2} + \gamma \left( \frac{1}{r}, M(y + \beta) \right) & \text{for } y > \exp(\mu) - \beta.
\end{cases}
\]

(128)
where

\[ B(y) = \left( \frac{\mu - \log(y + \beta)}{\sigma} \right)^r, \quad M(y) = \left( \frac{\log(y + \beta) - \mu}{\sigma} \right)^r, \]

and

\[ \Gamma(s, z) = \int_0^\infty e^{-t} t^{v-1} dt, \quad \gamma(v, z) = \int_0^z e^{-t} t^{v-1} dt \]

are the incomplete gamma functions.

### A.3 Changing Supply Estimation Equations Derivation

We impose \( \zeta = \zeta_1 = \zeta_2 \). In a very similar fashion to section ?? the following equations determine the quality supply in the 3 periods.

\[
R_1(h) = \int_0^h g_1(v_1(x)) dx
\]

(129)

\[
R_2(h) = \frac{1}{N_2} \int_0^h g_1(v_1(x)) \left( \frac{v_2(x)}{v_1(x)} \right)^{P_1 \zeta_1} dx
\]

(130)

\[
R_3(h) = \frac{1}{N_3} \int_0^h q_3(x) dx = \frac{1}{N_3} \int_0^h q_2(h) \left( \frac{v_3(x)}{v_2(x)} \right)^{P_2 \zeta} dx
\]

(132)

\[
= \frac{1}{N_3} \int_0^h q_1(x) \left( \frac{v_2(x)}{v_1(x)} \right)^{P_1 \zeta} \left( \frac{v_3(x)}{v_2(x)} \right)^{P_2 \zeta} dx
\]

\[
= \frac{1}{N_3} \int_0^h g_1(v_1(x)) \left( \frac{v_2(x)}{v_1(x)} \right)^{P_1 \zeta} \left( \frac{v_3(x)}{v_2(x)} \right)^{P_2 \zeta} dx
\]

where

\[
N_3 = \int_0^\infty q_3(x) dx = \int_0^\infty q_2(x) \left( \frac{v_3(x)}{v_2(x)} \right)^{P_2 \zeta} dx
\]

\[
= \int_0^h g_1(v_1(x)) \left( \frac{v_2(x)}{v_1(x)} \right)^{P_1 \zeta} \left( \frac{v_3(x)}{v_2(x)} \right)^{P_2 \zeta} dx
\]

For estimation, \( l^N_1(\psi) \) term becomes

\[
l^N_1(\psi) = \sum_{j=1}^{1} \left( G_3(v_3(h_j; \psi)) - R_3(h_j; \psi) \right)^2 + \left( G_2(v_2(h_j; \psi)) - R_2(h_j; \psi) \right)^2 + \left( G_1(v_1(h_j; \psi)) - R_1((h_j; \psi)) \right)^2
\]

(133)

where \( G_1(v_1(h_j; \psi)) = R_1((h_j; \psi)) \) by assumption.
A.4 obtaining renter’s data

The AHS report includes housing costs as a variable that groups together both rent and utilities for the households whose utilities are not included in the rent, and only rent for those who have them included. This is similar in other surveys that have rent data. To obtain renter’s data we calculate the Fraction of rent that is utility cost for the landlord (rucl)

$$rucl = \frac{re * me + rg * mg + ro * mo}{rm}$$

where $re$ is renters who are electricity users and have cost included in rent, $rg$ renters who are gas users and have cost included in rent, $ro$ renters who are oil users and have cost included in rent, $me$ is the median payment for electricity, $mg$ is the median payment for gas, $mo$ is the median payment for oil, and $r$ is the number of renters. The value of rent for each bin $b$ will be will be $rent_b = livingcosts_b * (1 - rucl/mr)$ where $mr$ is median rent.

A.5 parametric bootstrapping.

$N$ is the number of households in the random observed sample.

1. CLT: We assume that the estimated probabilities are N-distr, unbiased with $\sigma^2$ variance:

$$\sqrt{N}(\hat{p}_j^N - p_j) \sim N(0, \sigma^2)$$

where $\sigma^2 = p_j(1 - p_j)$.

2. For vectors $p = (P_1, ..., P_J)'$, $p = (P_1, ..., P_J)'$ : where $J$ is the number of quantiles observed in the data, we have

$$\sqrt{N}(\hat{p}_j^N - p_j) \sim N(0, \Sigma^2)$$

with $\Sigma_{jj} = p_j(1 - p_j), \Sigma_{ij} = -P_iP_j$. Using $\hat{p}_j^N$ we estimate $\hat{\Sigma}^N$

3. Algorithm:

   a) Get $\hat{p}_j^N$ and $\hat{\Sigma}^N$ from the observed quantiles in the data.

   b) Simulate a dew $p_s$ Use the asymptotic distribution $N(\hat{p}_j^N, \hat{\Sigma}^N)$

4. Estimate the model to get estimated parameters $\hat{\Psi}_S$ using the drawn $P_s$ as empirical distribution.

5. Repeat the algorithm $S$ times.
A.6 ELASTICITIES

A.6.1 Income Demand Elasticity

To derive elasticities for $t = 1$ we can assume housing is homogeneous and continuously divisible as we will use a linear pricing function for this period:

$$v = ph$$

Utility is

$$U = \alpha \ln(1 - \phi(h + \eta)^\gamma) + \ln(y - ph - \kappa)$$

FOCs are:

$$\frac{-\alpha \phi \gamma (h + \eta)^{\gamma - 1}}{1 - \phi(h + \eta)^\gamma} = \frac{p}{y - ph - \kappa}$$
$$-\alpha \phi \gamma (y - ph - \kappa) = p((h + \eta)^{1 - \gamma} - \phi(h + \eta))$$
$$-\alpha \phi \gamma(y - ph - \kappa) = p(h + \eta)((h + \eta)^{-\gamma} - \phi)$$
$$-\alpha \phi \gamma(y - \kappa) = -\alpha \phi \gamma h + p(h + \eta)((h + \eta)^{-\gamma} - \phi)$$

$$\frac{-\alpha \phi \gamma}{p} = -\alpha \phi \gamma h + (h + \eta)((h + \eta)^{-\gamma} - \phi)$$

$$y = \kappa - \frac{p}{\alpha \phi \gamma} [-\alpha \phi \gamma h + (h + \eta)^{1 - \gamma} - \phi(h + \eta)]$$

$$\frac{dy}{dh} = -\frac{p}{\alpha \phi \gamma} \left[-\alpha \phi \gamma + (1 - \gamma)(h + \eta)^{-\gamma} - \phi\right]$$

$$\frac{dh}{dy} = -\frac{\alpha \phi \gamma}{p} \left[\frac{1}{-\alpha \phi \gamma + (1 - \gamma)(h + \eta)^{-\gamma} - \phi}\right]$$

And the elasticity is

$$\frac{dh \ y}{dy \ h} = -\frac{\alpha \phi \gamma}{p} \left[\frac{1}{-\alpha \phi \gamma + (1 - \gamma)(h + \eta)^{-\gamma} - \phi}\right]$$

$$\kappa - \frac{p}{\alpha \phi \gamma} \left[-\alpha \phi \gamma h + (h + \eta)^{1 - \gamma} - \phi(h + \eta)\right]$$

$$\frac{1}{\frac{\alpha \phi \gamma}{h}}$$

Notice the income elasticity in this case depends on the price $p$. We are evaluating it at $p = 1$, which is the price we are assuming when we impose a linear pricing function $v(h) = h$.

For $t > 1$

The pricing function is

$$v_t(h) = \left(A_t \left[1 - (1 - \phi(h + \eta)^\gamma \alpha(b_t - 1)\right]\right)^{\frac{1}{\alpha(b_t - 1)}} - \theta_t$$
and its derivative is
\[
\nu'(h) = \frac{1}{1 - b_t} \left( A_t \left[ 1 - (1 - \phi(h + \eta)^\gamma)^{\alpha(b_t - 1)} \right] \right)^{\frac{1}{1 - \alpha}} - A (1 - \phi(h + \eta)^\gamma)^{\alpha(b_t - 1)} - \gamma \phi(h + \eta)^\gamma
\]

Utility is
\[
U = \alpha \ln(1 - \phi(h + \eta)^\gamma) + \ln(y - v(h) - \kappa)
\]

FOCs are:
\[
\frac{-\gamma \phi(y + \eta)^{\gamma - 1}}{1 - \phi(h + \eta)^\gamma} + \frac{-\nu'(h)}{y - v(h) - \kappa} = 0
\]
\[
y - v(h) - \kappa = \nu'(h) \frac{1 - \phi(h + \eta)^\gamma}{-\gamma \phi(y + \eta)^{\gamma - 1}}
\]
\[
y = \nu'(h) \frac{1 - \phi(h + \eta)^\gamma}{-\gamma \phi(y + \eta)^{\gamma - 1}} + v(h) + \kappa
\]

Taking derivatives
\[
\frac{dy}{dh} = \frac{A'B - B'A}{B^2} + \nu'(h)
\]
with
\[
A = \nu'(h)(1 - \phi(h + \eta)^\gamma)
\]
\[
A' = \nu''(h)(1 - \phi(h + \eta)^\gamma) + \nu'(h)(-\gamma \phi(y + \eta)^{\gamma - 1})
\]
\[
B = (-\gamma \phi \alpha(y + \eta)^{\gamma - 1})
\]
\[
B' = (-\gamma \phi \alpha(y - 1)(y + \eta)^{\gamma - 2})
\]

And the elasticity is
\[
\frac{dh}{dy} \frac{y}{h} = \frac{dh}{dy} \frac{\nu'(h)}{-\gamma \phi(y + \eta)^{\gamma - 1}} + v(h) + \kappa
\]
with \(\frac{dh}{dy}\) as above.

### A.6.2 Price Demand Elasticity

\[
p = \frac{-\alpha \phi \gamma (y - \kappa)}{-\alpha \phi \gamma h + (h + \eta)^{1-\gamma} - \phi(h + \eta)}
\]
\[
\frac{dp}{dh} = \frac{(-\alpha \phi \gamma (y - \kappa)) (-1)S'(h)}{S(h)^2}
\]
where \( S(h) = -\alpha \phi \gamma h + (h + \eta)^{1 - \gamma} - \phi(h + \eta), \) and \( S'(h) = -\alpha \phi \gamma + (1 - \gamma)(h + \eta)^{-\gamma} - \phi. \)

The elasticity is

\[
\frac{\partial h}{\partial p} = \frac{S(h)^2}{(-\alpha \phi \gamma (y - \kappa))} \frac{1}{(-1)S'(h)} \frac{1}{h} - \frac{\alpha \phi \gamma (y - \kappa)}{-\alpha \phi \gamma h + (h + \eta)((h + \eta)^{-\gamma} - \phi)}
\]
HETEROGENEOUS HOUSEHOLDS APPENDIX

B.1 ESTIMATION DETAILS: TYPES DEFINED BY K-MEANS CLUSTERING.

B.1.1 The Rent-to-Value ratios and equilibrium quality distribution

As we discussed in detail in previous sections, we can estimate the rent-to-value functions for each time period using our non-parametric matching estimator.¹

![Value Cdfs by type (New York 1999)](image)

![Value Cdfs by type (New York 2003)](image)

![Rent Cdfs by type (New York 1999)](image)

![Rent Cdfs by type (New York 2003)](image)

Figure 34: House values and rents distributions for each household type obtained through k-means clustering.

We use the estimated user costs in figure (36) to link values and rents in figure (35). This user cost can be used to transform the equilibrium hedonic pricing function into an equilibrium asset pricing function at different quality levels in figure (both in figure 38) It is also interesting to notice user costs by type that are aggregated to

¹ Note that data limitations make it difficult to estimate the locus outside a range of 50 and 350 thousand dollars for the 1995 and 2002, and 50 and 500 dollars in 2007.
obtain the final user cost, shown in figure (37). The final user cost is dominated by the behavior of the households in type 2, which is a consequence of them occupying a larger share of the population. Type 1 shows much higher user costs when considered separately. This could be interpreted as a tendency to assigning a lower value to owning. The opposite happens with type 3. Finally, we do not see large changes across periods. This suggests that the two periods are comparable. Also, the values obtained are consistent with normal values of the variable that comprise the user cost as defined in section 3.2.2. Epple, Quintero, and Sieg [2013] study more extreme changes of the user cost that occurred in the period of large run up in values in 2007 in several metropolitan areas in the US.
B.1 estimation details: types defined by k-means clustering.

Figure 37: User costs for each household type obtained through k-means clustering.

### B.1.2 Characteristics of the quality index

We obtain an equilibrium distribution of the housing quality one dimensional index $h$. This index is intended to capture all dimensions of quality a housing unit provides to a household, including those that are obtained from the neighborhood. We present table 9 as a way to roughly illustrate how characteristics change for households who consume a higher $h$. The table shows the mean of some selected characteristics available in our data housing quality consumption in the ranges noted in the corresponding column.

We observe that most variables show the behavior we expect as quality grows. The index of adequate housing is built by the Census Bureau summarizing a long series of infrastructure characteristics and problems. As expected, the best grade is obtained by the units that are of the highest quality. The same happens with all the rest of variables related to housing infrastructure (first 6 variables). The only unexpected behavior comes from square footage, in which we do not observe a monotonically increasing behavior.

The next 4 variables capture neighborhood characteristics. The neighborhood rating is given by the surveyed housholder. Crime seems to decrease with quality, while police service is better in neighborhoods with higher quality housing units. However, surprisingly, the evaluation of the local public school is not monotonically increasing in quality. This is actually an assessment given by the householder surveyed as well, and it asks from them to evaluate the local public school comparing it to others in the area. This makes this variable difficult to compare across neighborhoods located far from each other.

Overall, these characteristics illustrate that the housing quality variable captures infrastructure and neighborhood services characteristics in the expected way. At the same time, some of the differences observed are not that large, which implies that there are many other characteristics captured by housing quality. Future work would use the quality index $h$ to separate the observed and the unobserved characteristics in orthogonal spaces, and to determine the relative importance of each one in the determination of the equilibrium overall qual-
Table 9: Mean housing and neighborhood characteristics for some quality ranges, for New York City for 1999-2003.

<table>
<thead>
<tr>
<th>Quality range</th>
<th>Percentile in 1999</th>
<th>Percentile in 2003</th>
<th>10% - 15%</th>
<th>8% - 21%</th>
<th>45% - 50%</th>
<th>53% - 60%</th>
<th>85% - 90%</th>
<th>86% - 91%</th>
</tr>
</thead>
<tbody>
<tr>
<td>sqr footage</td>
<td>1365.045</td>
<td>1220.326</td>
<td>(452.0771)</td>
<td>(223.3806)</td>
<td>(140.0405)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># rooms</td>
<td>4.016</td>
<td>4.096</td>
<td>(.222)</td>
<td>(.130)</td>
<td>(.176)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># bathrooms</td>
<td>1.186</td>
<td>1.111</td>
<td>(.0769)</td>
<td>(.0419)</td>
<td>(.0734)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># times heat broke</td>
<td>3.863</td>
<td>2.858</td>
<td>(1.012)</td>
<td>(1.879)</td>
<td>(2.329)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem with rodents (Yes = 1, No = 2)</td>
<td>1.699</td>
<td>1.732</td>
<td>(1.012)</td>
<td>(1.879)</td>
<td>(2.329)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adequate housing (1-3, 1 best)</td>
<td>1.329</td>
<td>1.273</td>
<td>(0.070)</td>
<td>(0.066)</td>
<td>(0.033)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Neighborhood rating (1-10, 10 best)</td>
<td>6.941</td>
<td>7.169</td>
<td>(0.378)</td>
<td>(0.194)</td>
<td>(0.142)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Neighborhood crime (Yes=1, No=2)</td>
<td>1.672</td>
<td>1.669</td>
<td>(0.068)</td>
<td>(0.055)</td>
<td>(0.029)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Police satisfactory (Yes=1, No=2)</td>
<td>1.156</td>
<td>1.104</td>
<td>(0.050)</td>
<td>(0.035)</td>
<td>(0.019)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>School satisfactory (Yes=1, No=2)</td>
<td>1.277</td>
<td>1.111</td>
<td>(0.107)</td>
<td>(0.075)</td>
<td>(0.069)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### B.1 estimation details: types defined by k-means clustering.

The estimated parameters imply a good fit of the observed distributions, which is an important part of our estimation approach. The bands show a 95% confidence interval in the estimation of each point in the distribution calculated with the standard errors obtained through bootstrapping (see section B.3). Our discrete method estimates the price of each point in the partition of quality, and therefore provides different standard error for each price estimated.

#### Robustness: Chicago 1999, 2003

The figures in this section show a summary of the results of the model estimated for the Chicago metropolitan area, for 1999 and 2003. We see that the rates of households that rent are overall lower when compared with New York. The results for the 3 types determined by k-means clustering are similar with the case of New York. We observe some differences in the implied equilibrium elasticities can be partially explained by the difference in the equilibrium aggregate rents.
across types. However, we still see similar preference for housing quality across groups reflected in figure (42).

### B.1.4 Fit to the data

### B.1.5 Estimation Details: Types defined by the presence of children.

The market user costs that we use to aggregate renters and owners in the case of types being defined by the presence of children are the same as in the section above. If we calculate user costs separately by type, we also get similar results, with user costs for the households with no children located above those with children. Similar conclusions are reached when comparing the differences in user costs in this case than when we compared them between households of type 1 and 2 from the k−means defined types. In contrast, type 3, although is comprised of households with few or no children, is behaving in a closer manner to the households with children in the current analysis. Just like before, we use the user cost to link equilibrium price (rent) hedonic functions and value hedonic function in figure (47). Figure (48) shows we get a good fit.

### B.1.6 Robustness: Chicago 1999, 2003

The figures in this section show a summary of the results of the model estimated for the Chicago metropolitan area, for 1999 and 2003. We see a similar behavior in the income and equilibrium aggregate rents distributions. As a consequence, figure (49) shows that the patterns of preference for housing quality across types is robust in this metropolitan area when compared to New York. The different obtained price and income elasticities across types are similar as well (figure (50)).
Figure 40: Percentage of owners and renters for each household type obtained through k-means clustering. Chicago

B.2 Proofs

Proof 9 The single-crossing condition implies that there is stratification of households by income in equilibrium. Stratification implies that there exists a distribution function for house values $G_t(v)$ such that:

$$F_t(y) = G_t(v)$$

(137)

Hence there exists a monotonic mapping between income and housing value. If $F_t$ is strictly monotonic, it can be inverted, and hence $F_t^{-1}$ exists. Q.E.D.

Proof 10 We can write the household’s optimization problem as:

$$\max_h u_1(h) + u_2(y - v(h))$$

(138)

The FOC of this problem with respect to $h$ is given by:

$$u'_1(h) - u'_2(y - v(h)) v'(h) = 0$$

(139)
Now define $h^* = v(h)$ and hence $h = v^{-1}(h^*)$. The decision problem associated with this model is then

$$
\max_{h^*} u_1(v^{-1}(h^*)) + u_2(y - h^*)
$$

and the FOC with respect to $h^*$ is

$$
u'_1(v^{-1}(h^*)) v^{-1'}(h^*) - u'_2(y - h^*) = 0
$$

Now $h = v^{-1}(h^*) = v^{-1}(v(h))$ and hence $v^{-1'}(h^*) v'(h) = 1$. Hence we conclude that the two models are observationally equivalent. In the first case, we have non-linear pricing and in the second case we have linear pricing. Q.E.D.

**Proof 11** Given our normalizations, we have also identified the housing supply function in the first period since $R_1(h) = G_1(v)$ which then identifies the density of housing quality in the first period $q_1(h)$. Consider a continuous setting.

Proposition implies that $v_2(h)$ is identified. As a consequence $G_2(v_2(h))$ is identified. Moreover, $V_1(h)$ and $V_2(h)$ are observed by the econometrician. As a consequence $\zeta$, is identified of the market clearing condition:

$$
R_2(h) = k_2 \int_0^h q_1(x) \left( \frac{V_2(x)}{V_1(x)} \right)^\zeta \, dx
$$

Q.E.D.

Note that this proof generalizes for more complicated parametric forms of the supply function.
B.3 parametric bootstrap

N is the number of households in the random observed sample.

1. CLT: We assume that the estimated probabilities are \( N \)-distr, unbiased with \( \sigma^2 \) variance:

\[
\sqrt{N}(\hat{p}_j^N - p_j) \sim \mathcal{N}(0, \sigma^2) \quad (143)
\]

where \( \sigma^2 = p_j(1 - p_j) \).

Figure 43: Share of income using types obtained through k-means clustering. Chicago.
Figure 44: Elasticities using types obtained through k-means clustering.

2. For vectors $p = (P_1, ..., P_J)'$, $p = (P_1, ..., P_J)'$ : where $J$ is the number of quantiles observed in the data, we have

$$\sqrt{N}(\hat{p}_j^N - p_j) \sim N(0, \Sigma^2) \quad (144)$$

with $\Sigma_{jj} = p_j(1 - p_j), \Sigma_{ij} = -P_iP_j$. Using $\hat{p}_j^N$ we estimate $\hat{\Sigma}^N$.

3. Algorithm:
   a) Get $\hat{p}_j^N$ and $\hat{\Sigma}^N$ from the observed quantiles in the data.
   b) Simulate a dew $p_s$. Use the asymptotic distribution $N(\hat{p}_j^N, \Sigma^N)$.

4. Estimate the model to get estimated parameters $\hat{\Psi}_S$ using the drawn $P_s$ as empirical distribution.

5. Repeat the algorithm $S$ times.
Figure 45: User costs for each household with types defined by the presence of children.

Figure 46: House values and rents distributions for each households with and without children.

Figure 47: Estimated pricing functions using types defined by the presence of children.
Figure 48: Fit of aggregate rent distributions with confidence intervals using types defined by the presence of children.
Figure 49: Income, aggregate rents distribution, and share of income and income loci using types defined by presence of children. Chicago.
Figure 50: Elasticities using types defined by the presence of children. Chicago.
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DECLARATION

Put your declaration here.

Darmstadt, April 2014

Luis Eduardo Quintero