Essays on Sponsored Search Advertising

by

AMIN SAYEDI

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Dissertation Committee:
Kinshuk Jerath (Co-Chair)
R. Ravi (Co-Chair)
Tim Derdenger
Isa Hafalir
Kannan Srinivasan

Tepper School of Business
Carnegie Mellon University
Abstract

My dissertation examines the strategic interactions of search engines and firms in sponsored search advertising. In my first essay, I study how a firm decides to allocate its advertising budget between sponsored search advertising and traditional channels of advertising. An advantage of sponsored search advertising is that, since the firm advertises in response to a consumer-initiated search, the sales-conversion rate is typically higher than in display advertising. However, a disadvantage of sponsored search advertising is that, to “steal” the firm’s customers, its competitors can also bid on its keyword, therefore driving the advertising costs higher. Using a game theory model, I study the implications of these tradeoffs on the advertising decisions of competing firms, and on the design of the sponsored search auction by the search engine. I find that symmetric firms may follow asymmetric budget allocation and bidding strategies. Moreover, the search engine benefits from discouraging competition in sponsored search auctions by shielding firms from competitors’ bids. This explains the practice of employing “keyword relevance scores”, under which search engines such as Google, Yahoo! and Bing under-weight the bids of firms bidding on competitors’ keywords. I also obtain various other interesting insights on the interplay between sponsored search advertising and traditional advertising and support the results by short-term and long-term data collected on poaching behavior of firms in several industries.

Prior auction theory literature has paid relatively little attention to budget constraints. In the second essay in my dissertation, motivated by sponsored search advertising, I introduce a mechanism for budget-constrained advertisers. The mechanism, a generalization of the Vickrey auction, is near-Pareto optimal in ex-post Nash equilibrium. Furthermore, understating budgets or values is weakly dominated. Since revenue is increasing in budgets and values, all kinds of equilibrium deviations from true valuations turn out to be beneficial to the auctioneer in the proposed mechanism.

In the third essay of my dissertation, I look at “exclusivity contracts” in sponsored search advertising. As sponsored search becomes increasingly important as an advertising medium for firms, search engines are exploring more advanced bidding and ranking mechanisms to increase their revenues from sponsored search auctions. For instance, Microsoft, Yahoo! and Google are investigating auction mechanisms in which each advertiser submits two bids: one bid for the current display format in which multiple advertisers are displayed, and one bid for being shown exclusively. If the exclusive-placement bid by an advertiser is high enough then only that advertiser is displayed, otherwise multiple advertisers are displayed and ranked based on their multiple-placement bids. I study a natural modification of the
GSP mechanism that Yahoo! has recently proposed. I show that although allowing the advertisers to bid for exclusivity can generate higher revenue for the search engine, under certain conditions, it can also create new class of equilibria with significantly lower revenue. Allowing exclusivity might be better or worse for the advertisers depending on their values for exclusivity as well as the heterogeneity in their values for exclusivity. Finally, when exclusive bidding is allowed, even if exclusive display is not the outcome, the search engine can extract higher revenue because allowing two bids increases the competition among the advertisers.
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Online advertising is the fastest growing channel of advertising, likely to exceed 25% of the total US advertising expense, by 2015. Firms spend the largest portion of their online advertising budget on sponsored search advertising, which captures nearly half of the share. In sponsored search advertising, advertisers pay a fee to the search engine to have links to their websites listed as relevant results in response to a keyword search. When a user submits a query on the search engine, she is presented with advertisements (henceforth, ads) that are placed into positions, usually arranged linearly down the side of the page (along with the organic search results which are not sponsored). Sponsored search advertising is the primary source of revenue for search engines; for instance, Google, Yahoo! and Bing earn millions of dollars per day through this channel.

Since the environment of sponsored search is relatively new, current solutions and mechanisms suffer from deficiencies as search engines and advertisers continue to improve their mechanisms and advertising strategies. Search engines have to decide how to sell their advertising space to advertisers while advertisers have to decide upon their advertising strategies in sponsored search advertising along with other channels of advertising.

Being their largest source of revenue, the pricing mechanism for sponsored search advertising is of critical importance to search engines. All the prominent search engines currently run Generalized Second Price (GSP) auctions to sell their advertising space. However, this choice of the auction mechanism was not a straightforward one, and the industry went through several phases before the GSP auction became the dominant choice. Sponsored search was introduced in 1997 when the search engine company GoTo (renamed Overture in 2001, and acquired by Yahoo in 2003) let advertisers bid to appear among the top search results. GoTo’s original sponsored search mechanism was a Generalized First Price auction in which every advertiser submitted a bid and the advertisers were arranged in descending order of bids, with each one paying its bid. This auction was also adopted by both Yahoo! and Bing.

and MSN (now rebranded as Bing). The payment mechanism was also experimented with, and while initially advertisers had to pay every time their ad was shown (pay per impression), this was changed to payment every time their ad was actually clicked (pay per click). The Generalized First Price auction, however, was soon found to be an unstable auction mechanism in which advertisers had the incentive to cyclically bid low and high amounts to game the system. This motivated the need for a more stable mechanism.

In 2002, Google introduced the Generalized Second Price (GSP) auction with the basic rules that every advertiser submits its per-click bid (i.e., how much it is willing to pay for every click obtained from a consumer) but has to actually pay only the minimum amount necessary to keep its current position in the list of results (i.e., GSP is a “next-price” auction). GSP, a much more stable auction mechanism, was gradually adopted by other prominent search engines as well (e.g., Yahoo! and Bing currently also use this mechanism). Search engines also continually conduct their own internal experimentation, based on which they apply slight variations (the exact details of which are often not publicly announced) to the basic GSP mechanism. For instance, advertisers are now ranked based not on their submitted bids, but based on their effective bids, where each advertiser’s effective bid is obtained by multiplying its submitted bid with a quality score specific to the advertiser-keyword combination. As another example, search engines now allow pay-per-action payment mechanism in which an advertiser has to pay only if, after clicking on an ad, a user also completes a predetermined action, such as purchasing the product or spending two minutes on the advertiser’s website.

The above discussion shows that the spectacular rise of sponsored search advertising in the last decade has been accompanied by constant effort from the search engines to refine their pricing mechanisms by gradually fixing the deficiencies in them, including developing new auction mechanisms to rank advertisers. Moreover, if one search engine introduces a profitable innovation, other search engines follow suit in a short time. As the industry matures, search engines are looking to further expand their bidding mechanisms by allowing advertisers to be more specific about their utilities and to express a richer set of preferences.

Adopting sponsored search advertising has not been easy for businesses either. While firms are progressively increasing sponsored search advertising expenditure, many managers are “confused” with sponsored search. A recent study by American Express (2011) shows that more than half of small business owners say that they need help with search engine marketing. In a survey of more than 1000 small businesses, Opus Research (2008) found that “confusion” is the main reason that firms are not doing online marketing. Even the businesses who use sponsored search may be doing it inefficiently. Clickable (2008) shows that majority of small and mid-size businesses fail to track sponsored search effectiveness due to complexity.

The source of managers’ confusion is not hard to see given the above discussion on fast changes in sponsored search and its rapid growth. While details are never revealed, search engines announce several updates in their advertising auctions every few months. Furthermore, new businesses join sponsored search and competitors change their strategies in response to each other. All of these make sponsored search even more difficult for firms to manage. Finally, share of sponsored search from total US advertising expenditure is growing fast, and is predicted (eMarketer (2012)) to reach 15% by 2016. This creates new challenges for businesses as they have to analyze implications of sponsored search on other marketing
decisions like promotions and pricing as well.

1.1 Thesis Contribution and Results

In my dissertation, I study several challenges faced by search engines and advertisers in regard to sponsored search advertising. From search engine’s point of view, I discuss how it can alter their selling mechanism to improve total welfare and revenue, and how changes in the selling mechanism affect advertisers’ behavior. From advertiser’s point of view, I study the interactions of sponsored search advertising and other channels of advertising in a competitive setting. Specifically, in the first chapter of my dissertation, I study how a firm decides to allocate its advertising budget between sponsored search advertising and traditional channels of advertising. An advantage of sponsored search advertising is that, since the firm advertises in response to a consumer-initiated search, the sales-conversion rate is typically higher than in display advertising. However, a disadvantage of sponsored search advertising is that, to “steal” the firm’s customers, its competitors can also bid on its keyword, therefore driving the advertising costs higher. Using a game theory model, I study the implications of these tradeoffs on the advertising decisions of competing firms, and on the design of the sponsored search auction by the search engine. I find that symmetric firms may follow asymmetric advertising strategies, with one firm focusing on traditional advertising and the other firm focusing on sponsored search with poaching. Interestingly, the search engine benefits from handicapping poaching, i.e., it benefits from discouraging competition in its own auctions. This explains why search engines such as Google, Yahoo! and Bing use “keyword relevance” scores to under-weight the bids of firms bidding on competitors’ keywords. I also obtain various other interesting insights on the interplay between sponsored search advertising and traditional advertising and support the results by short-term and long-term data collected on poaching behavior of firms in several industries.

In the second chapter in my dissertation, I study the implications of advertisers’ budget constraints on search engines’ mechanism design and, in turn, on advertisers’ bidding strategies. In a setup where a divisible good, corresponding to total supply of clicks for a period of time, is to be allocated to a set of bidders with budget constraints, I introduce a mechanism in the spirit of the Vickrey auction. In the mechanism I propose, understating budgets or values is weakly dominated. Moreover, the revenue is increasing in budgets and values; therefore, all kinds of equilibrium deviations from true valuations turn out to be beneficial to the auctioneer. I also show that ex-post Nash equilibrium of the mechanism is near Pareto-optimal in the sense that all full winners’ values are above all full losers’ values.

In the third chapter of my dissertation, I study how expressiveness and bidding language can affect search engines’ revenue, advertisers strategies and social welfare. As sponsored search becomes increasingly important as an advertising medium for firms, search engines are exploring more advanced bidding and ranking mechanisms to increase their revenues from sponsored search auctions. For instance, Microsoft, Yahoo! and Google are investigating auction mechanisms in which each advertiser submits two bids: one bid for the current display format in which multiple advertisers are displayed, and one bid for being shown exclusively. If the exclusive-placement bid by an advertiser is high enough then only that advertiser is displayed, otherwise multiple advertisers are displayed and ranked based on
their multiple-placement bids. I study two modifications of the GSP mechanism that Yahoo! has recently proposed. I show that allowing the advertisers to bid for exclusivity can generate higher revenue for the search engine; however, it also creates new equilibria some of which may lead to lower revenue for the search engine. Allowing exclusivity might be better or worse for the advertisers depending on their values for exclusivity as well as the heterogeneity in their values for exclusivity. Finally, when exclusive bidding is allowed, even if exclusive display is not the outcome, the search engine still extracts higher revenue because allowing two bids increases the competition among the advertisers.

Sponsored search advertising consists of a game between advertisers, competing for clicks, a game between advertisers and search engine, fighting on price of a click, and a game between search engines, competing for innovation. In the first essay of my thesis, I focus on competition between firms. I show how this competition can affect other marketing decisions of firms as well as search engines’ mechanism design. In the second essay, I focus on the game between advertisers and a search engine where advertisers’ goal is to get maximum number of clicks at minimum price and search engine’s goal is to increase revenue and efficiency. Finally, in the third essay I study search engines’ innovations for future sponsored search. I discuss the advantages and disadvantages of a new auction being investigated by major search engines for implementation. I conclude the dissertation by a high-level discussion of the results and future trends of research in sponsored search advertising.
Chapter 2

Competitive Poaching on Sponsored Search Advertising and Strategic Impact on Traditional Advertising

2.1 Introduction

Online advertising is the fastest growing channel of advertising, likely to exceed 30% of the total US media advertising expenditure by 2015 (Online Marketing Trends 2012). This rapid growth in online advertising is impressive given that television advertising, which firms have been using for decades, has a market share of about 35%. On aggregate, firms allocate nearly half of the online advertising spend to sponsored search advertising (eMarketer 2012). There are several unique advantages of sponsored search advertising, including effective targetability, ease of setting up a campaign and ease of measurement of ROI. Given its unique characteristics and spectacular growth, sponsored search advertising is received increasing attention from researchers and practitioners. However, while firms are dedicating progressively larger fractions of their advertising budgets to sponsored search advertising at the expense of traditional channels of advertising, the strategic implications of the interactions between these two types of advertising have not been carefully researched.

Consumers go through several stages of involvement before purchasing a product, and different types of advertising influence consumers differently in these stages. A widely employed marketing framework that captures the various sequential stages of a typical consumer’s decision process before final purchase is the awareness-interest-desire-action (AIDA) model. Traditional channels of advertising, such as television, newspapers, radio and billboards, generate awareness among consumers and are located more towards the initial stages of the AIDA model. In traditional advertising, contact with potential consumers is initiated by the firm to make them aware of and interested in the firm’s brand or product. Sponsored search, however, is located more towards the last stages of the AIDA model and it influences a consumer close to the purchase action. Sponsored search effectively targets the consumers who are already aware of the product and have shown some interest or desire in the product by searching for an associated keyword at a search engine. In the context of the AIDA model, traditional advertising can be interpreted as “upstream advertising,” while sponsored search...
can be interpreted as “downstream advertising.” Thus, traditional awareness-generating advertising and sponsored search advertising are inter-related and play complementary roles in successfully consummating the sale of a product.

In a strategic market with competing firms, creating awareness has benefits as well as perils, especially when the awareness created through traditional advertising for one brand can be exploited by sponsored search advertising by a competing firm. Competitors, instead of allocating their advertising budget to create awareness about their own products, can advertise in sponsored search on the keywords of a firm in the same industry that is creating awareness by investing in traditional advertising, trying to steal the latter’s potential customers. We refer to this as “poaching” in sponsored search.

We provide anecdotal examples where such poaching is evident. The shoe company Skechers advertised its Shape Ups model during Super Bowl 2011 (held on February 6, 2011). The upper panel of Figure 2.1(a) shows the effect of the television ad on the search volumes of the keywords “Skechers” and “Shape Ups” on Google in the days after Super Bowl. It can be easily seen that the advertising created considerable awareness resulting in heavy keyword search on the internet. Interestingly, while Skechers spent millions of dollars for the Super Bowl commercials, its competitor, Reebok, poached on the keyword “Shape Ups” to advertise its competing model called “Easy Tone,” as shown in the screenshot of a Google search for the keyword “Shape Ups” in the lower panel of Figure 2.1(a). The same phenomenon can be observed for the keyword “Groupon”—the company Groupon advertised on Super Bowl which led to a significant increase in the search volume of its associated keyword (upper panel of Figure 2.1(b)), while its competitor LivingSocial poached on its keyword (lower panel of Figure 2.1(b)). These are only a few of many instances of poaching, which is happening with increasing frequency on the internet.

In summary, poaching happens when a firm creates awareness resulting in pertinent keyword search on the internet, and competing firms aggressively bid on these keywords to display their ads. In fact, since the competitors are spending less to create awareness, they can bid more aggressively on sponsored search keywords (typically sold through position auctions run by the search engine) and thus can even enjoy an advantage over the firm that has attracted the customers in the first place. Such poaching behavior has implications not only for the competing firms’ strategies on the sponsored search and traditional advertising channels, it also strategically affects the search engine’s auction strategy. In this chapter, we examine these issues using an analytical framework. We address three broad questions. First, under what conditions will poaching arise and be beneficial for a firm? Second, what are the effects of poaching on competing firms’ decisions for budget allocation among traditional and sponsored search advertising? Third, what are the consequences of poaching for the search engine and how should it adapt its auction mechanism anticipating poaching by competing advertisers.

We first consider the case in which there are two identical competing firms. We find the existence of an asymmetric equilibrium in which one firm mostly advertises on traditional channels and creates awareness, while its competitor poaches on its keyword in sponsored search. This is because poaching increases the competition in sponsored search auctions which increases the advertising prices of the keywords. This increases the per-customer acquisition cost to the firm from sponsored search, which makes sponsored search a less desired option, and incentivizes the firms to move a larger share of their money to traditional
Figure 2.1: Poaching
advertising. However, poaching remains a profitable strategy for one firm, which leads to the asymmetric budget allocation. Interestingly, although the competition in sponsored search increases with poaching, the search engine’s revenue may decrease because of the incentive of firms to move some of their advertising budget to traditional advertising. As a result, the search engine may benefit from discouraging competition in its own keyword auctions by making poaching harder for the firms. This offers an explanation for why major search engines such as Google, Yahoo! and Bing use “keyword relevance scores” to under-weight the bids of firms bidding on competitors’ keywords.

We extend our analysis to the case of asymmetric firms with different advertising budgets. When the difference between budgets is large enough, the firm with the smaller advertising budget has a greater incentive to poach as compared to the firm with the larger budget, because the latter conducts more traditional advertising and drives more traffic towards its keywords. Interestingly, with asymmetric firms, the search engine may in fact benefit from poaching—its revenue is maximized when the poaching is controlled but not prohibited. Knowing that the smaller-budget firm has a large incentive to poach, the search engine can employ appropriate keyword relevance scores to under-weight the poaching bid and make it more expensive for the smaller-budget firm to poach, but only to the extent that it still continues to poach. This serves a dual purpose: first, it helps the search engine to capture a large part of the advertising budget of the smaller-budget firm through poaching; second, it shields the larger-budget firm by controlling bid escalation due to poaching to some extent, which means that this firm does not move much of its budget away from sponsored search.

From this point of view, keyword relevance measures could be interpreted as a complex price discrimination mechanism: for the smaller-budget firm, poaching is a very desirable option; for this reason, the search engine can charge it firm a higher price than the larger-budget firm which is creating the search volume. This result explains why search engines are in support of allowing bidding on trademarked keywords by competitors (Parker 2011), yet still employ keyword relevance measures to handicap poaching firms. We also present several extensions of our model to show the robustness of our results.

A growing theoretical and empirical literature on sponsored search advertising has enhanced our understanding of its different aspects; this includes Athey and Ellison (2009), Chan and Park (2010), Desai et al. (2011), Ghose and Yang (2009), Jerath et al. (2011), Katona and Sarvary (2010), Rutz and Bucklin (2011), Yang and Ghose (2010), Yao and Mela (2009) and Zhu and Wilbur (2011). Our work is distinctly different from the above work because they consider sponsored search advertising in isolation while we model it in a multichannel advertising setting. Goldfarb and Tucker (2011a,b) study substitution between online and offline advertising induced by better targeting in online advertising and advertising bans on offline advertising for certain products. Joo et al. (2011) empirically show that television advertising increases Internet search volume; we use their finding as a building block in our model. Kim and Balachander (2010) model sponsored search in a multichannel setting. However, they do not consider poaching behavior of competing firms, and the resulting strategic response of the search engine (in terms of auction design). In our research, the analysis of these two aspects leads to a rich set of results and insights which have anecdotal support. Finally, Bass et al. (2005) show the existence of free-riding in traditional advertising where one firm focuses on generic advertising to expand the market and its competitor focuses on brand advertising to steal market share. We also show the
existence of free-riding effects, due to poaching, in our model. However, our study is very different from theirs because in sponsored search firms can target which customers to free ride on (for instance, by poaching only those who search a competitor’s keyword), which is not possible in the scenario they consider. Moreover, we have the search engine as a strategic agent in our model, an element which has no parallel in their model.\footnote{1}

The rest of the chapter is structured as follows. In Section 4.3, we describe the model. In Section 2.3, we analyze the model with symmetric firms to develop key insights, and in Section 2.4, we extend to the case of asymmetric firms. In Section 2.5, we analyze “keyword relevance scores” as a strategic device used by a search engine to control poaching. In Section 2.6, we consider several extensions of the basic model and show that the key insights are unchanged under each extension, while we obtain additional interesting results. In Section 2.7, we present some empirical and anecdotal evidence supporting some of our results. Finally, in Section 2.8, we conclude with a discussion and lay out some possible directions for future research.

\section{Model}

Our model consists of three entities: the firms, the users, and the search engine. We start with a simple model with symmetric firms to communicate some key concepts and insights, and later proceed to the case of asymmetric firms. Two firms, Firm 1 and Firm 2, produce identical products. Each firm has an exogenously specified total budget $B$ allocated for advertising, and has to decide how to allocate its advertising budget to traditional advertising and sponsored search advertising to maximize total sales. For Firm $i$, we denote the money spent on traditional advertising by $T_i$ and the money spent on sponsored search by $S_i$, where $T_i + S_i = B$. Note that we bundle all non-sponsored search channels of advertising together into traditional advertising.

As discussed earlier, we focus on the awareness-creating aspect of traditional advertising. When Firm $i$ spends $T_i$ on traditional advertising, it generates awareness for its product $\footnote{1}{We note that “downstream advertising,” where the aim is to reach customers expected to have a high likelihood of purchase, is also possible in certain channels other than sponsored search. For instance, firms may advertise in yellow pages to reach customers when they are specifically looking for the provider of a product or service before making a purchase. However, targetability is weak in yellow pages which makes it difficult to poach a competitor’s customers; for instance, among the customers who are consulting yellow pages, firms cannot distinguish between those who are interested in a competitor versus those who are already interested in the firm itself. Similarly, “checkout coupons” used in retail stores, powered by technology from providers such as Catalina Marketing, target customers based on their profiles (purchase history, gender, location, etc.). This allows targeting consumers who purchased a competitor’s product in a category (Pancras and Sudhir 2007). However, in this case, the identification of the customer and subsequent targeting is after the current purchase is made (with the idea of making the customer switch at the next purchase occasion), which makes poaching less effective. Sponsored search, on the other hand, makes for a unique combination of features that make it an extremely effective channel for poaching—based on the keyword searched consumers self-identify whether they are interested in the competitor, the firm itself, or the category; the consumers can be targeted after they are interested in the product but still before the purchase; based on the keyword searched, different consumers can be targeted differently by showing them different ad copies and landing pages upon clicking.}{2}{We make the budget endogenous in Section 2.6.5 and confirm that the results of our basic model are robust.}}$
among \((1 + \alpha)T_i\) customers, where \(\alpha > 0\). These customers either buy the product directly or search for the product at a search engine. Each firm is associated with a specific keyword which consumers use to search for it on the search engine. For instance, if Apple sells the iPad and Samsung sells the Galaxy Tab, then the keywords associated with Apple and Samsung are “iPad” and “Galaxy Tab,” respectively. For simplicity, we assume that there is only one advertising slot available for each keyword, i.e., only one firm is shown in response to a keyword search. This simplification does not impact the insights from the model. When a customer searches for a keyword, the search engine uses a pay-per-click second-price auction with exogenous reserve price \(R\) to sell the advertising slot for that keyword to the firm that bids higher. However, when a consumer clicks on the sponsored link, the winner has to pay the loser’s bid or the reserve price, which ever is higher.

The transaction of a customer who searches the product is either influenced by the sponsored links or not. It is not important in our model whether a customer purchases directly from the firm after being exposed to a traditional ad or searches but ignores the sponsored search results and then purchases from the firm. Therefore, without loss of generality, we assume that all the customers who search the product are influenced by sponsored search results. Specifically, we assume that out of the \((1 + \alpha)T_i\) customers made aware by traditional advertising, \(\alpha T_i\) customers buy the product independent of what they see in sponsored search and the remaining \(T_i\) customers carry out a search for Firm \(i\)’s keyword at a search engine and purchase the product that they see in the sponsored search section of the results (which may or may not be Firm \(i\)’s product). The scaling assumptions above basically imply that the ratio of the number of transactions that were not influenced by sponsored search to those that were influenced is \(\alpha\).

Note that out of the \((1 + \alpha)T_i\) customers who are exposed to traditional advertising, only \(\alpha T_i\) make a purchase directly. The remaining \(T_i\) customers overflow to the sponsored search channel and all of them purchase from the firm whose ad they see in response to their search. Therefore, the customers who are exposed to traditional advertising and are “upstream” in the AIDA framework have a smaller purchase-conversion rate (conversion rate equal to \(\alpha/(1 + \alpha)\)) than the customers who are “downstream” in the AIDA framework and are exposed to sponsored search advertising (conversion rate equal to 1). Consequently, there is a trade-off between traditional advertising and sponsored search. On the one hand, the firm can decide to create awareness through traditional advertising and obtain some direct sales. On the other hand, the firm may choose to advertise on the competitor’s keyword in sponsored search and rely on the awareness created by the competitor.

**Key Intermediate Result**

Due to each firm’s traditional advertising, \(T_i\) customers search keyword \(i\) at the search engine. These customers arrive sequentially at the search engine and it runs a separate auction for each customer. In other words, the search engine sequentially runs \(T_i\) auctions for each

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3We later consider the extension in which there is a third keyword which is the category keyword, such as “tablet” in the above example.

4Note that we are implicitly assuming that advertising response function is linear; however, our results apply to other response functions proposed in the literature such as concave and S-shaped functions as well. Details of this analysis are available upon request.
keyword. In each auction, the firms submit their bids simultaneously. Each firm decides its bid in an auction based on the budget it has allocated for the keyword and how much of it is remaining when a specific customer arrives. Using subgame-perfect equilibrium, we show in Theorem 6.1.5 in the appendix that the unique outcome of this sequential second-price auction coincides with the outcome of a market-clearing-price mechanism. We state this result in the lemma below.

**Lemma 2.2.1.** Assume that Firm 1 spends $L_1$ and Firm 2 spends $L_2$ on keyword $i$ searched by $T_i$ consumers. If $L_1 + L_2 \geq T_iR$ then $L_1/(L_1 + L_2) \cdot T_i$ customers purchase from Firm 1, and $L_2/(L_1 + L_2) \cdot T_i$ customers purchase from Firm 2. If $L_1 + L_2 < T_iR$ then $L_1/R$ customers purchase from Firm 1 and $L_2/R$ customers purchase from Firm 2.

This result is interesting in itself and, to the best of our knowledge, is new to the auctions literature. This is also a very useful result as, for the analysis in the rest of the chapter, it allows us to reduce bidding in a complicated sequential auction to a much simpler form that abstracts away from the auction and, in fact, represents a simple market-clearing allocation. In the analysis to follow, rather than modeling bidding between competitors in each and every scenario, we will simply use this lemma repeatedly.

**Defining the Firms’ Strategies**

In general, a firm’s strategy is any splitting of advertising budget between the traditional channel, its own keyword in sponsored search, and the competitor’s keyword in sponsored search. For simplicity, we restrict the strategy space of the firms to three strategies, each focusing on one of the three channels. Specifically, we allow a firm to follow one of the following three pure strategies. A firm’s strategy can also be a mixed strategy, meaning that each of the strategies below will be played with a certain probability.

1. Own Keyword Focus (Own): The firm focuses on its own keyword in sponsored search advertising, trying to target the consumers who are in late stages of their purchase processes.

2. Traditional Focus (Traditional): The firm focuses on traditional advertising, trying to create awareness and interest about the product.

3. Poaching Focus (Poach): The firm focuses on the competitor’s keyword in sponsored search advertising, trying to steal potential customers of the competitor.

Let $T^O$, $T^T$ and $T^P$ be the amount of money that a firm spends on traditional advertising when using the Own, Traditional and Poach strategies, respectively; the superscripts $O$, $T$ and $P$ stand for Own, Traditional and Poach respectively. We now consider these strategies one by one.

---

5Note that if $R > 1/\alpha$, then it is a strictly dominant strategy for the firms to spend all of their budget on traditional advertising. In other words, if the reserve price is so high that the cost of attracting a customer in sponsored search is higher than the cost in traditional advertising, the firms should spend all of their budgets on traditional advertising in any strategy. In reality, this situation is unlikely to happen because it means that the search engine has set the reserve price of sponsored search advertising so high that no advertiser wants to advertise through sponsored search.
In the Own strategy, the firm focuses on its own keyword. A natural definition would be to assume that the firm spends all of its budget for advertising on its own keyword in sponsored search. However, this implies that the keyword will have zero search volume because nothing has been spent on awareness-generating traditional advertising, which implies that there will be no revenue. In other words, even when the firm wants to “maximally focus” on its own keyword, it should not spend all of its budget on its keyword in sponsored search, and should transfer some of its budget to traditional advertising to generate the necessary search volume for its keyword. In fact, we will define the budget allocation in the Own strategy in such a way that any strategy that allocates more budget to the firm’s own keyword as compared to the Own strategy will always be weakly dominated by the allocation of the Own strategy. In this sense, the “Own Keyword Focus” strategy has “maximal focus” on the firm’s own keyword. We now derive this allocation.

According to the second-price auction of the search engine, the firm has to pay at least $R$ per customer in sponsored search advertising. And according to the model, if it spends $T_i$ on traditional advertising, $T_i$ customers would search the product. Therefore, the firm has to spend at least $T^O = \frac{B}{R+1}$ on traditional advertising even if it wants to focus on sponsored search advertising of its own keyword. Consequently, the firm spends $B - T^O = \frac{BR}{R+1}$ on its own keyword when using Own strategy. In general, it can be proved that spending more than $B - T^O$ for sponsored search advertising of own keyword is weakly dominated by spending $B - T^O$.

In the Traditional strategy, the firm focuses on traditional advertising. Similar to the Own strategy, we can show that spending all of its budget on traditional advertising is a dominated strategy. As before, we define the Traditional strategy in a way that the firm has “maximal focus” on traditional advertising, i.e., under no conditions should the firm have an incentive to allocate more to traditional advertising than what it allocates in this strategy. Suppose that the firm has budget $B$ and its competitor has budget $B'$. In the Traditional strategy, the amount of budget that the firm spends on traditional advertising is defined as $T_T = \min(B, \frac{B+B'}{R+1}, B + B' - \sqrt{\frac{B'(B+B')}{1+\alpha}})$. It can be proved that spending more than $T_T$ on traditional advertising is weakly dominated by spending $T_T$ on traditional advertising. In other words, no matter what strategy the competitor uses, the return to the firm when spending $T_T$ on traditional advertising is always greater than or equal to the return it gets when spending more than $T_T$ on traditional advertising. Consequently, in the Traditional strategy, the firm spends $B - T_T$ for sponsored search advertising on its own keyword.

Finally, in the Poach strategy, the firm spends all of its budget for sponsored search advertising on the competitor’s keyword. Hence, we have $T_P = 0$.

It is critically important to note here that the above allocation of budget is not something that the players of the game are doing as part of the game. On the contrary, we as researchers are defining the strategies to set the strategy space such that the pure strategies are undominated, i.e., when a firm is following a strategy of focusing on one of the three forms of advertising, the allocation is such that the firm spends maximum amount of budget on that form of advertising but still uses an undominated strategy. When the players play the game, they will play one of these pure strategies, or play a mixed strategy.

Also note that, for simplicity, we are not considering any pure strategies other than the above three. In Section 6.1.5 in the appendix, we consider a more general model in which we
allow firms to choose any arbitrary allocation of budget among the three forms of advertising considered (traditional advertising, advertising on its own keyword in sponsored search, and poaching by advertising on the competitor’s keyword in sponsored search). We show that the results and insights obtained in our basic model here are not affected. Specifically, we show that, under mild conditions on the parameters, the equilibria of this simpler model are also equilibria of the more general model, and there are no other equilibria in the more general model. This equivalence highlights the appropriateness of the strategy space chosen here. We continue to use the simpler model because the key insights are easier to communicate with this model, and it is easier to incorporate extensions into it.

The order of moves in the model is as follows. First, the search engine announces the rules of the auction (that it is a second-price, pay-per-click auction). Second, the two firms simultaneously decide their budget allocation strategies. Third, consumers see traditional ads and a fraction $\alpha/(1 + \alpha)$ of them buy directly from the firm whose ad they saw. Fourth, the remaining consumers go to the search engine sequentially and search the keyword of the firm whose traditional ad they saw, and the sequential second-price auction is played out. Fifth, each consumer who searches, purchases from the firm that is shown to her in the sponsored search results.

Finally, note that we have assumed the price of the product to be exogenous. We make this choice to focus solely on competition between firms in the sponsored search auction, and not confound it with price competition. In Section 2.6.7, we allow for price competition as well and confirm that the insights we obtain from our basic analysis hold.

2.3 Analysis with Symmetric Firms

In this section, we examine the case of symmetric firms to develop basic insights into the problem.

2.3.1 Revenue Analysis

We use $\Pi^{i,j}$, where $i, j \in \{O, T, P\}$, to denote the revenue of a firm that is using strategy $i$ while its competitor is using strategy $j$. For example, $\Pi^{P,O}$ denotes the revenue of a firm playing the Poach strategy whose competitor is playing the Own strategy. Note that the firms are sales maximizers.

According to the definition of the Own strategy, the revenue of a firm that is playing the Own strategy, as long as its competitor does not poach, will be $\Pi^{O,O} = \Pi^{O,T} = T^O(1 + \alpha)$. However, if the competitor poaches, by Lemma 2.2.1 its revenue will be $\Pi^{O,P} = T^O(\alpha + \frac{B - T^O}{2B - T^O})$. The search engine’s revenue from a firm that is playing the Own strategy will be $B - T^O$.

According to the definition of the Traditional strategy, the revenue of a firm that is playing the Traditional strategy, as long as the other firm does not poach, will be $\Pi^{T,O} = \Pi^{T,T} = \alpha T^T + \min(\frac{B - T^T}{R}, T^T)$. However, if the competitor poaches, by Lemma 2.2.1 its
revenue will be $\Pi_{T,P} = \alpha T + T^T \left( \frac{B - T^T}{2B - T^T} \right)$. The search engine’s revenue from a firm that is playing the Traditional strategy will be $B - T^T$.

Now consider a firm that is playing the Poach strategy. If the competitor also poaches simultaneously, no money is spent on traditional advertising and hence no customer is gained. Therefore, the revenue of both firms will be zero, $\Pi_{P,P} = 0$. However, if the competitor plays the Own strategy, the revenue of the poaching firm will be $\Pi_{P,O} = T^O \frac{B}{2B - T^O}$. If the competitor plays the Traditional strategy, the revenue of the poaching firm will be $\Pi_{P,T} = T^T \frac{B}{2B - T^T}$. Notice that a firm should not play Poach if the competitor is playing Poach (because $\Pi_{P,P} = 0$, which is less than both $\Pi_{O,P}$ and $\Pi_{T,P}$). Similarly, a firm should not play Poaching if the competitor is playing Own (because $\Pi_{P,O} < \Pi_{O,O}$). We find that the only way that a firm can benefit from playing Poach is if the competitor plays Traditional, i.e., $\Pi_{P,O} < \Pi_{O,O}$. Furthermore, playing Poach can be beneficial only if $\Pi_{P,T} > \Pi_{O,T}$ (note that $\Pi_{O,T} > \Pi_{T,T}$ already holds), which gives the following condition:

$$R > \frac{1}{1 + 2\alpha} \left( \sqrt{2(1 + \alpha)^3} - \alpha \right).$$  \hspace{1cm} (2.1)

The above condition implies that poaching is profitable only if $R$ is large enough. Intuitively, if $R$ is small, the firm finds it more profitable to conduct its own traditional advertising and capture the customers that overflow into search at the cheap reservation price, thus avoiding competition with the other firm. However, as $R$ becomes larger, the customers from sponsored search do not come cheap any more. When $R$ is large enough, the firm finds it more profitable to free ride on the awareness generation of the other firm (i.e., not spend anything from its own budget on awareness generation) and, in fact, use all of its advertising budget to compete with the other firm in the auction. Finally, the search engine’s revenue from a firm playing Poach will always be $B$ unless the other firm is also playing Poach, in which case the search engine’s revenue will be 0.

2.3.2 Equilibrium Analysis

Given the three pure strategies for each firm, we obtain the two-person normal-form game depicted in Table 2.1.

<table>
<thead>
<tr>
<th>Firm 1</th>
<th>Poach</th>
<th>Own</th>
<th>Traditional</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poach</td>
<td>$(\Pi_{P,P}, \Pi_{P,P})$</td>
<td>$(\Pi_{P,O}, \Pi_{O,P})$</td>
<td>$(\Pi_{P,T}, \Pi_{T,P})$</td>
</tr>
<tr>
<td>Own</td>
<td>$(\Pi_{O,P}, \Pi_{O,P})$</td>
<td>$(\Pi_{O,O}, \Pi_{O,O})$</td>
<td>$(\Pi_{O,T}, \Pi_{T,O})$</td>
</tr>
<tr>
<td>Traditional</td>
<td>$(\Pi_{T,P}, \Pi_{T,P})$</td>
<td>$(\Pi_{T,O}, \Pi_{O,O})$</td>
<td>$(\Pi_{T,T}, \Pi_{T,T})$</td>
</tr>
</tbody>
</table>

Table 2.1: Payoff matrix of the firms’ strategies

**Nash Equilibria:** The game has both pure- and mixed-strategy Nash equilibria. Since $\Pi_{O,O} \geq \Pi_{P,O}$ and $\Pi_{O,O} \geq \Pi_{T,O}$, both firms using Own strategy is always a pure-strategy Nash equilibrium. If the reserve price $R$ is large enough such that the inequality in (2.1) holds, since $\Pi_{P,T} \geq \Pi_{O,T} \geq \Pi_{T,T}$ and $\Pi_{T,P} \geq \Pi_{O,P} \geq \Pi_{P,P}$, one firm using Poach and the other firm using Traditional is also a pure-strategy Nash equilibrium; otherwise, it is not.
Note that, even though the two firms are symmetric, the above is an asymmetric equilibrium in which one firm spends all of its budget on poaching while the other firm spends a larger portion of its budget on traditional advertising (as compared to the case when the competitor is not poaching).

The equilibria of the game always conform to the following pattern. When $R$ is small, only the (Own, Own) equilibrium is obtained in which no firm is poaching. As $R$ increases and the inequality in (2.1) holds, two new types equilibria arise. One is the asymmetric (Poach, Traditional) equilibrium discussed above (and its counterpart, the (Traditional, Poach) equilibrium). The third type of equilibrium is a mixed-strategy equilibrium in which one firm mixes between Poach and Own, and the other firm mixes between Traditional and Own. Note that the mixed-strategy equilibrium is also an asymmetric equilibrium. As $R$ increases further and becomes larger than $1/\alpha$, both firms allocate all their budget to traditional advertising, as explained earlier.

Figures 2.2(a) and 2.2(b) denote the revenues of firms. For clarity in the graphs for the cases with asymmetric equilibria, we designate one firm as the “poaching firm” (this firm always poaches in the (Poach, Traditional) equilibrium and mixes between Poach and Own in the mixed equilibrium) and the other firm as the “Traditional firm” (this firm always uses Traditional in the (Poach, Traditional) equilibrium and mixes between Traditional and Own in the mixed equilibrium). We can observe from Figures 2.2(a) and 2.2(b) that when poaching equilibria exist, the poaching firm’s revenue is higher than in the non-poaching (Own, Own) equilibrium, while the non-poaching firm’s revenue is lower. In other words, in this asymmetric equilibrium, the non-poaching firm is accommodating the poaching firm’s free-riding behavior.

**Search Engine’s Revenue:** Different equilibria of the game have different revenue expressions for the search engine. Table 2.2 summarizes the search engine’s revenue in each of the outcomes.

<table>
<thead>
<tr>
<th>Firm 1</th>
<th>Poach</th>
<th>Own</th>
<th>Traditional</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poach</td>
<td>0</td>
<td>$2B-T^O$</td>
<td>$2B-T^T$</td>
</tr>
<tr>
<td>Own</td>
<td>$2B-T^O$</td>
<td>$2B-2T^O$</td>
<td>$2B-T^O-T^T$</td>
</tr>
<tr>
<td>Traditional</td>
<td>$2B-T^T$</td>
<td>$2B-T^O-T^T$</td>
<td>$2B-2T^T$</td>
</tr>
</tbody>
</table>

Table 2.2: Search engine’s payoff matrix

The search engine’s revenue is depicted in Figure 2.2(c). First, note that this revenue increases in $R$ until $R = 1/\alpha$ where it drops to zero. Second, there is no poaching for low values of $R$. For high values of $R$, multiple equilibria exist and the search engine’s revenue is the same from the (Own, Own) and the (Poach, Traditional) equilibria, which is larger than the revenue from the mixed equilibrium. Since there is a positive likelihood of the low-revenue mixed-strategy equilibrium existing, this implies that the existence of poaching may lower the search engine’s revenue for high values of $R$ (even though poaching is increasing competition in the search engine’s auction). This happens because the Traditional firm allocates more of its advertising budget to traditional advertising. The following proposition summarizes this important result.
Proposition 2.3.1. Symmetric firms may follow asymmetric budget allocation strategies in which one firm allocates a larger fraction of its advertising budget to poaching on its competitor in sponsored search advertising, while the other firm allocates a larger fraction of its advertising budget to awareness-generating traditional advertising. Furthermore, the search engine’s revenue may decrease in the presence of poaching.

2.4 Asymmetric Firms with Different Advertising Budgets

In this section, we assume that the firms may have different advertising budgets. Without loss of generality, through scaling, we assume that the budget of the smaller-budget firm is 1, and the budget of the larger-budget firm is \( B \geq 1 \). For expositional simplicity, from this point on we call the smaller-budget firm as the “weak” firm, denoted by subscript the \( W \), and the larger-budget firm as the “strong” firm, denoted by the subscript \( S \).

Definitions of Strategies

In this asymmetric firms case, we rederive the budget allocations for the Own, Traditional and Poach strategies for the strong and weak firms. As in Section 4.3, the budget allocation is based on the idea that when a firm is focusing on one of the three options (namely, Own, Traditional and Poach), the allocation is such that the firm spends the maximum amount of budget on that option but still uses an undominated strategy. Accordingly, for the Own strategy, we have \( T^O_W = \frac{1}{R+1} \) and \( T^O_S = \frac{B}{R+1} \). For the Traditional strategy, we have
\[
T^T_W = 1 + B - \sqrt{\frac{B(1+B)}{1+\alpha}}, \quad \text{if} \quad \frac{B+1}{R+1} \geq 1 + B - \sqrt{\frac{B(1+B)}{1+\alpha}}, \quad \text{and} \quad T^T_W = \frac{B+1}{R+1}, \quad \text{otherwise.}
\]
Similarly, we have
\[
T^T_S = 1 + B - \sqrt{\frac{1+B}{1+\alpha}}, \quad \text{if} \quad \frac{B+1}{R+1} \geq 1 + B - \sqrt{\frac{1+B}{1+\alpha}}, \quad \text{and} \quad T^T_S = \frac{B+1}{R+1}, \quad \text{otherwise.}
\]

Note that we define “smaller” and “larger” budget with respect to the campaigns that the firms are running at a given time, not their total advertising budgets. For instance, in the Skechers and Reebok example in the introduction, around the time of Super Bowl 2011, Skechers spent millions of dollars to advertise on TV during Super Bowl and generate internet traffic for its keyword while Reebok did not; therefore, in this case Skechers would be the larger-budget firm even though Reebok may have a larger annual advertising budget.
Poach strategy, we have $T_W^A = T_S^A = 0$.

**Revenue Analysis**

For the Own strategy, if no firm poaches on the keyword of the other, the situation is very similar to the symmetric case. Particularly, we have $\Pi^{O,O}_W = T_W^O(1 + \alpha)$ and $\Pi^{O,T}_S = T_S^O(1 + \alpha)$. If the strong firm poaches, $\Pi^{O,P}_W = T_W^O(\alpha + \frac{1-T_W}{B+1-T_W})$. Similarly, if the weak firm poaches $\Pi^{P,O}_S = T_S^O(\alpha + \frac{B-T_S}{1+B-T_S})$. For the Traditional strategy, the revenue of the weak firm, if the other firm poaches, is $\Pi^{T,P}_W = \alpha T_T^W + T_W^T \frac{1-T_W}{B+1-T_W}$. However, if the strong firm does not poach, the revenue of the weak firm is $\Pi^{T,O}_W = \Pi^{T,T}_W = \alpha T_T^W + \min(T_T^W, \frac{1-T_T}{R})$. The revenue of the strong firm is $\Pi^{T,P}_S = T_S^T \alpha + (\frac{B-T_S}{1+B-T_S})T_T^S$, and $\Pi^{T,O}_S = \Pi^{T,T}_S = \alpha T_T^S + \min(T_T^S, \frac{B-T_T}{R})$. For the Poach strategy, for the strong firm, we have $\Pi^{P,P}_S = 0$, $\Pi^{P,O}_S = T_S^O(\frac{B}{B+1-T_W})$ and $\Pi^{P,T}_S = T_S^T(\frac{B}{B+1-T_W})$. For the weak firm, we have $\Pi^{P,P}_W = 0$, $\Pi^{P,O}_W = T_W^O(\frac{1}{1+B-T_S})$, $\Pi^{P,T}_W = T_W^T(\frac{1}{1+B-T_S})$.

**Equilibrium Analysis**

We solve the game with asymmetric firms fully analytically. We provide the results and insights here; the derivations are provided in Section 6.1.1 in the appendix. The results in the previous section with symmetric firms can be obtained by substituting $B = 1$ in the solution to follow.

We use the following terminology for brevity. When describing equilibria, we assume that the first firm is the weak firm, and the second firm is the strong firm. For example, by (Poach, Traditional) equilibrium we mean an equilibrium in which the weak firm uses Poach and the strong firm uses Traditional. We will also see two types of mixed equilibria. In the first mixed equilibrium, which we call the Weak-Poach-mixed equilibrium, the weak firm mixes between Poach and Own, and the strong firm mixes between Traditional and Own. In the second mixed equilibrium, which we call the Strong-Poach-mixed equilibrium, the weak firm mixes between Traditional and Own, and the strong firm mixes between Poach and Own.

The different equilibria that arise for different values of the strong firm’s budget $B$ and the reserve price $R$ can be understood by jointly examining Table 2.3 and Figure 2.3. Based on the different sets of equilibria that arise in different regions of the parameter space, there are four regions in Figure 2.3, labeled $A$, $B$, $C$ and $D$; these equilibria are shown in the second column of Table 2.3. Within each region, there are sub-regions based on whether poaching increases or decreases the search engine’s profit. The third column of the table indicates the impact of poaching on the search engine’s profit in that region. The letter $Y$ means that poaching can increase the search engine’s profit, the letter $N$ means that poaching decreases the search engine’s profit and $Y/N$ means that poaching can increase or decrease the search engine’s profit depending on equilibrium selection.

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More precisely, $Y$ means that the set of equilibria with poaching equilibria weakly dominates (from the search engine’s perspective) the set of equilibria without poaching equilibria. $N$ means that set of equilibria
Figure 2.3: Poaching regions and search engine’s revenue for different levels of budget asymmetry and different values of reserve price.

$R^W, R^S, R^*, R^{**}, R_m,$ and $\bar{R}$, and analytical definitions of low, medium and high asymmetry, are available in Section 6.1.1 in the appendix.

From the above results, we find that the weak firm poaches in all regions except region $A$ while the strong firm poaches only in region $B$, i.e., the weak firm poaches in a larger parameter space as compared to the strong firm. We also show that the relative gain from poaching of the weak firm is larger than that of the strong firm (i.e., $\frac{\Pi^{P,T}_W}{\Pi^{O,O}_W} \geq \frac{\Pi^{P,T}_S}{\Pi^{O,O}_S}$). Moreover, the weak firm’s incentive to poach increases with increasing budget asymmetry (i.e., $\frac{\Pi^{P,T}_W}{\Pi^{O,O}_W}$ is an increasing function of $B$). This is intuitively because the strong firm has a relatively large search volume; therefore, the poaching of the weak firm does not affect the sponsored search price significantly, and in turn, allows poaching at a relatively low price. In fact, if the firms are very asymmetric, the incentive to poach is so high that (Poach, Traditional) is the only equilibrium of the game (region $D$). We state this in the proposition below.

$\text{without poaching equilibria weakly dominates the set of equilibria with poaching equilibria. Y/N means that the sets of equilibria with and without poaching equilibria cannot be compared, i.e., depending on equilibrium selection, either option may have higher revenue for the search engine. See the appendix for the definition of “weak dominance” to compare sets of equilibria. Also note that for region $D$ (Poach, Traditional) is the unique equilibrium, so we can trivially say that poaching increases search engine’s revenue. For region $A$, we cannot make such a comparison since there is no poaching equilibrium, and we use the symbol “-” to denote this.}$
<table>
<thead>
<tr>
<th>Region</th>
<th>Equilibria</th>
<th>Poaching Beneficial for Search Engine</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$(O, O)$</td>
<td>$-$</td>
</tr>
</tbody>
</table>
| $B$    | $(O, O), (P, T), (T, P)$, Weak-Poach-mixed, Strong-Poach-mixed | $B1$: Y/N  
$B2$: N |
| $C$    | $(O, O), (P, T)$, Weak-Poach-mixed | $C1$: Y  
$C2$: Y/N  
$C3$: N  
$C4$: Y |
| $D$    | $(P, T)$   | Y                                    |

Table 2.3: Description of the regions in Figure 2.3

(a) Poaching region  
(b) Fraction of budget allocated to traditional advertising, by the firm being poached, as a function of $B$, for $\alpha = 3$  
(c) Fraction of budget allocated to traditional advertising, by the firm being poached, as a function of $\alpha$, for $B = 10$

Figure 2.4: Poaching region and the strategy of the firm being poached.

**Proposition 2.4.1.** When the advertising budget of one firm is larger than the advertising budget of the other firm, the lower-budget firm has a larger incentive to poach on the higher-budget firm, and the higher-budget firm’s best response is to accommodate this poaching rather than to retaliate.

In the case of budget asymmetry, poaching may be beneficial for the search engine. This is intuitively because the weak firm can only steal a small fraction of the strong firm’s customers. As a result, the strong firm’s response to the weak firm’s poaching is not as significant as in the case of symmetric firms, i.e., the strong firm does not shift a lot of its budget from sponsored search to traditional advertising in response to poaching. On the other hand, the weak firm spends all of its money on sponsored search. Hence, the search engine’s revenue may increase in the presence of poaching. We state this in the proposition below.

**Proposition 2.4.2.** If the asymmetry in the advertising budgets of firms is large enough, poaching can increase the search engine’s revenue.

We now examine the firms’ strategies as functions of $B$ (the budget asymmetry between firms) and $\alpha$ (the relative effectiveness of traditional advertising) for given exogenous $R$.  

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The plot in Figure 2.4(a) is representative of the regions in which poaching occurs in the B-α space (for reserve price small enough). Two interesting observations can be made from this figure. First, for a given level of α, poaching happens only if budget asymmetry is large enough. Intuitively, poaching becomes more attractive for a firm as its competitor’s search volume becomes larger. Second, for a fixed level of B, poaching does not happen if α is large enough, i.e., if the proportion of the consumers who are not influenced by sponsored search is large enough. Intuitively, if the proportion of the consumers who are influenced by sponsored search is small, trying to compete for and poach on those consumers is not a good strategy. These two observations are summarized in the following proposition.

**Proposition 2.4.3.**

(a) For a given proportion α of consumers who are not influenced by sponsored search advertising, poaching happens only if the budget of one firm is enough larger than the budget of the other firm.

(b) For a given level of budget asymmetry B between the firms, poaching happens only if the proportion of consumers who are influenced by sponsored search advertising is large enough.

Figure 2.4(b) and Figure 2.4(c) show the strong firm’s budget allocation to traditional advertising as function of B and α, respectively. Because of the existence of multiple equilibria (poaching and non-poaching), there are two curves depicting the advertising strategy of the firm, each corresponding to one equilibrium. In Figure 2.4(b), if B < 3.90, there is no poaching, if 3.90 ≤ B < 9.33, both poaching and non-poaching equilibria exist, and if B ≥ 9.33, only the poaching equilibrium exists. Interestingly, the strong firm’s strategy is not monotone in B. The jump in the percentage of budget allocated to traditional advertising in the poaching equilibrium is because the strong firm switches to the Traditional strategy from the Own strategy. After the jump, we see that the percentage gradually decreases as B increases. This is because as the strong firm’s budget increases, it is hurt less by poaching, and its response to poaching is moderated.

In Figure 2.4(c), if α < 3.35, only the poaching equilibrium exists, if 3.35 ≤ α < 9.03, both poaching and non-poaching equilibria exist, and if α > 9.03, there is no poaching. Note that within each regime (poaching or non-poaching), the percentage of budget allocated by the stronger firm to traditional advertising increases with α, as expected. However, when switching from the poaching to the non-poaching regime, the percentage of budget that the strong firm allocates to traditional advertising suddenly drops, because the strong firm changes its strategy from Traditional to Own. However, the percentage again increases as α increases further.

### 2.5 Keyword Relevance Measures

All the major search engines, including Google, Yahoo! and Bing, transform an advertiser’s submitted bid into an effective bid before determining the outcome of the sponsored search

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Mathematically, for poaching to happen we need $\Pi_{W}^{O,O} \leq \Pi_{W}^{P,T}$. In other words, poaching must be more profitable than the Own strategy for the weak firm to poach, given that the strong firm uses Traditional. It is easy to see that $\Pi_{W}^{P,T} - \Pi_{W}^{O,O}$ is increasing in B.

Mathematically, this can be verified by observing that $\Pi_{W}^{P,T} - \Pi_{W}^{O,O}$ is decreasing in α.
A multiplier is typically used to compute the effective bid, and this multiplier depends on many parameters including the advertiser’s past performance in terms of the click-through-rate (the probability that a customer who sees the advertiser’s sponsored link clicks on the link), the quality and reputation of the advertiser’s product or website, and the relevance of the keyword being bid on to the advertiser. Our focus here is on the last parameter and we explain it using the example below.

Consider the keyword “iPad” and the two firms Apple and Samsung. Apple is much more relevant to this keyword than Samsung, since Apple produces the iPad while Samsung only sells a competing product, namely Galaxy Tab, in the same category (tablets). Therefore, if the relevance of Apple to the keyword “iPad” is 1 on a scale from 0 and 1, the relevance of Samsung to this keyword is less than 1 and is, say, 0.5. For simplicity, assume that both firms have the same scores on other parameters used to calculate the multiplier for calculating the effective bid (click-through-rate, quality reputation, etc.). Suppose that Apple bids $1 per click and Samsung bids $1.5 per click to be displayed in response to the keyword “iPad.” It seems natural that the search engine should prefer to display Samsung instead of Apple in this case (assuming only one is displayed) as Samsung should generate more revenue than Apple for it. However, surprisingly, in a situation like this, the search engine decides to display Apple because of higher relevance to the keyword. In fact, Samsung will have to bid and pay at least $1/0.5 = $2 to win this keyword. If Samsung bids $1.5, Apple wins the keyword and has to pay only $0.75 per click.

One explanation for the existence of “relevance measures” is that the search engine wants to improve user experience by showing ads most directly relevant to the keyword searched by the users. Although this is a reasonable explanation, we argue that it is probably not the only explanation. We provide an alternative explanation—a search engine may use keyword relevance measures to handicap poaching selectively and to the extent it wants, and increase its revenue in the process. To simplify and focus on the effect of relevance measures, we assume that both firms have the same click-through rate, and the same quality and website reputation; this does not impact our results qualitatively.

We assume that if a firm wants to bid on the keyword of the other firm, its bid (i.e., the bid of the poaching firm) will be multiplied by $0 \leq \gamma \leq 1$. If $\gamma = 1$, we are in the framework that we have been in so far, i.e., firms poach on each others’ keywords without any handicap. On the other extreme, if $\gamma = 0$, firms cannot bid on each others’ keyword. This is similar to the situation where bidding on trademarked keywords is not allowed. We study the effect of intermediate values of $\gamma$ on the search engine’s revenue. To allow for asymmetric firms, we stay with the setting where one firm has budget $B \geq 1$ while the other firm has budget of 1.

Definitions of Strategies

As before, we first define the pure strategies—Own, Traditional and Poach—by deriving the budget allocations for strong and weak firms based on the condition that when a firm is following a strategy of focusing on one of the three types of advertising, the allocation is such that the firm spends maximum amount of budget on that form of advertising but still uses an undominated strategy. The Own and the Poaching strategies remain the same. However, the Traditional strategy changes slightly because the firm using Traditional strategy now knows that the bid of the Poaching firm is not as effective as it was when there was no poaching.
handicap. As a result, we have \( T_W^T = 1+\gamma B - \sqrt{\frac{\gamma B(1+\gamma B)}{1+\alpha}} \), if \( \frac{\gamma B+1}{R+1} \geq 1+\gamma B - \sqrt{\frac{\gamma B(1+\gamma B)}{1+\alpha}} \), and \( T_W^T = \frac{\gamma B+1}{R+1} \), otherwise. Similarly, \( T_S^T = \gamma + B - \sqrt{\frac{\gamma (\gamma + B)}{1+\alpha}} \), if \( \frac{B+\gamma}{R+1} \geq \gamma + B - \sqrt{\frac{\gamma (\gamma + B)}{1+\alpha}} \), and \( T_S^T = \frac{B+\gamma}{R+1} \), otherwise. The expressions are essentially the same as derived in Section 2.4, except that when the weak firm uses Traditional against the strong firm’s poaching it takes the strong firm’s budget as \( \gamma B \) instead of \( B \) and, similarly, when the strong firm uses Traditional against the weak firm’s poaching it takes the weak firm’s budget as \( \gamma \cdot 1 = \gamma \) instead of 1.

Revenue and Equilibrium Analysis

In this case, there is an additional initial stage in the game in which the search engine decides the value of \( \gamma \). After this stage, the analysis proceeds in the same way as in the previous cases, for which we construct and analytically solve the 3\( \times \)3 strategic-form game. We omit the details here and focus on the results, specifically, the value of \( \gamma \) chosen by the search engine and the resulting impact on poaching.

Figure 2.5 shows the optimal values of the relevance multiplier, labeled as \( \gamma^* \), chosen by the search engine for different values of budget asymmetry, \( B \). This is a representative plot, where the values of the other parameters are \( R = 1.5 \) and \( \alpha = 0.5 \). When the budget asymmetry is not too large, the search engine wants to prohibit poaching by the weak firm. It achieves this by setting \( \gamma \) small enough (i.e., the poaching penalty is large enough). The reason to prohibit poaching for low asymmetry is the same reasons as in Section 2.3—poaching in sponsored search makes it more expensive to obtain a customer through this channel, and in response the strong firm allocates more budget to the traditional advertising channels.

However, if \( B \) is large enough, the search engine sets \( \gamma \) at a medium value to maximize its profit, as indicated by the curve in the plot. In this case, poaching is penalized to some degree, but not so much that it does not happen. Intuitively, in spite of being penalized, the weak firm poaches (to free ride on the other firm’s awareness generation), in response to which the strong firm uses Traditional and moves more of its budget to traditional awareness-generating advertising. However, the “keyword relevance penalty” protects the strong firm to some extent by reducing the effective bid of the weak firm on its keyword, thus keeping bids from escalating too high. Shielded in this way, the strong firm does not need to shift
as large a portion of its money to the traditional channel (as it would have done if poaching were not penalized). However, at the same time, the weak firm has to pay a higher price per click and the search engine extracts all the budget of the weak firm. Interestingly, keyword relevance measures could be interpreted as a complex price discrimination mechanism. For the weak firm, poaching is a very desirable option; for this reason, the search engine can charge the weak firm a higher price than the strong firm which is creating the search volume.

**Proposition 2.5.1.** If the advertising budget of the strong firm is large enough compared to that of the weak firm, the search engine handicaps poaching by competitors but does not prohibit it.

To summarize, our basic model shows that firms in an industry have the incentive to poach in sponsored search, especially firms with relatively smaller advertising budgets. The best response of competitors is to accommodate this poaching behavior. Surprisingly, even though poaching leads to more competition in its auctions, under certain conditions the search engine has the incentive to handicap poaching. We now proceed to some extensions of the basic model.

## 2.6 Extensions

### 2.6.1 Category Keyword

In this extension, we assume that there is a category keyword which attracts customers from the traditional advertising of both firms. For example, in the context of tablets, “iPad” and “Galaxy Tab” are keywords specific to the companies Apple and Samsung, respectively, while “tablet” is a category keyword that describes both products. Some customers who see traditional awareness-generating ads of iPad or Galaxy Tab may search the keyword “tablet” instead of searching the product name. In accordance with this, we assume that some fraction of the customers who are exposed to traditional advertising of each firm search the category keyword instead of the firm-specific keyword. The insights obtained in Section 2.3 (asymmetric budget allocation strategies and reduction in search engine revenue due to poaching) also hold under this extension. More details are available in Section 6.1.2 in the appendix.

### 2.6.2 Reputation Effects

In the basic model, we assumed that a firm needs to advertise on the traditional channel to generate awareness and have non-zero search volume on the search engine. In this section, we drop this simplification and assume that a firm may have search volume even without recently-conducted awareness-generating advertising, say due to previous reputation. In other words, we assume that $V$ customers search a firm’s product even if it does not advertise on the traditional channel. We also let the firms to be asymmetric in this aspect by assuming that the reputation-based search volume of the “strong” firm is $V > 0$ while the “weak” firm has no reputation-based search volume (i.e., the weak firm’s keyword will be searched only if it does awareness-generating advertising).
The detailed analysis of this extension is included in Section 6.1.3 in the appendix. We confirm that the insights obtained from the basic model hold. Furthermore, if \( V \) is large enough (the strong firm has much larger reputation-based search volume than the weak firm), only the weak firm wants to poach, and its incentive to poach increases with \( V \). The strong firm, which already has high customer awareness, accommodates this poaching by spending more on traditional advertising. This gives us the following counter-intuitive result.

**Proposition 2.6.1.** The firm that has larger reputation-based customer awareness spends even more on awareness-generating advertising as a best response to the poaching of its competitor.

### 2.6.3 Display Advertising through the Search Engine

Internet display advertising is primarily an awareness-creating channel of advertising. Therefore, for the purpose of our modeling, we bundle Internet display advertising in the traditional channel (even though it has risen to prominence only in the last two decades). However, unlike other traditional advertising channels, Internet display advertising is largely controlled by the popular search engines that also control sponsored search advertising. Examples of such services are “Ad Exchange” by Google, “AdECN” by Microsoft and “Right Media” by Yahoo!

The above observation has an interesting implication—in response to poaching, a firm may move its money away from sponsored search, but spend some of this money on display advertising with the same search engine. In this case, the search engine still obtains the revenue, which might have different implications for its auction design strategy. We extend our model such that a fraction \( 0 < \delta < 1 \) of the money spent on traditional channel goes to the search engine. We find that this extension produces the same insights as in the basic model in Section 2.3.

### 2.6.4 Reduced Effectiveness of Poaching Advertisements

In the basic model, we assume that consumers purchase the product from the company whose ad they see in response to their keyword search, even if it is a poaching ad. However, it could be the case that if the name of the company or brand in the displayed ad does not match with the keyword searched, there is a smaller chance that the consumer will click on the link and convert after clicking. Another factor contributing to this could be that the organic results in response to the keyword search will have content related directly to the original keyword. If the consumer views the organic results as well before clicking on the sponsored link, then a poaching ad may reduce the consumer’s confidence in the poaching firm. For these and other reasons, a poaching ad may be less effective than a non-poaching ad. The reduced effectiveness of poaching ads, however, does not change our results qualitatively—while poaching will become a less desirable strategy, it will still be employed as a viable strategy by firms and all the associated insights will hold (unless the reduction in effectiveness is very severe, which is not always the case). Analytically, this is similar to the case in Section 2.5 in which the poaching firm is handicapped. We omit the details due to space constraints.
2.6.5 Endogenous Budget

In this section, we relax the assumption that the advertising budget of each firm is exogenously given, and allow each firm to decide how much to spend on advertising while trying to maximize its profit. Although there is no hard constraint on how much a firm can spend on advertising, we assume that spending more on advertising becomes harder as the firm spends more (Fernandez-Corugedo 2002), reflecting the fact that it is increasingly difficult to raise larger amounts of money. The profit of Firm \( i \) is

\[
\Pi_i = \alpha T_i + \min(T_i \frac{S_i}{S_i + P_j}, \frac{S_i}{S_i + P_j}) + \min(T_j \frac{P_i}{S_j + P_i}, \frac{P_i}{S_j + P_i}) - \eta(T_i + S_i + P_i)^\rho
\]

where \( j = 3-i \) represents the index of the other firm, \( T_i, S_i \) and \( P_i \), respectively, represent the level of advertising on traditional channel, sponsored search of own keyword, and sponsored search of competitor’s keyword. The parameter \( \rho > 1 \) captures the fact that increasing the advertising budget becomes harder as this budget becomes larger. Note that except for the budget term \( \eta(T_i + S_i + P_i)^\rho \), the profit expression is the same as the profit expression presented in Section 4.3.

By numerically calculating the equilibria of this revised formulation, we confirm that the results presented in Sections 2.3 and 2.4 are robust to budget endogeneity. Similar to the case of exogenous budget, we show that firms may use different advertising strategies in equilibrium. In particular, there exist asymmetric equilibria in which one firm focuses more on traditional advertising while its competitor poaches on its keyword. The slight difference from the exogenous budget setting is that symmetric firms with endogenous budget may poach on each others’ keywords at the same time. However, the degree of poaching could be different for the two firms with one firm poaching more than the other one. In addition, similar to Section 2.5, we show that the search engine’s revenue is maximized when poaching is slightly penalized. At a medium level of penalty, the firms accept the penalty and poach on each others’ keywords. Since poaching is a desirable option for each firm, they spend more on poaching to compensate the penalty imposed by the search engine, which increases the search engine’s revenue. However, if the penalty is too high, the firms’ best responses are not to poach. This can decrease the search engine’s revenue. Therefore, a medium level of penalty is where the search engine’s revenue is maximized.

2.6.6 Firms’ Decision Sequence

In the basic model in Section 4.3, we assumed that the firms decide how to allocate their budget to different channels of advertising simultaneously. However, one might argue that, in reality, the firms can observe each others’ traditional advertising efforts when deciding about sponsored search advertising. The results presented in Sections 2.3 and 2.4 are robust under this alternative decision sequence. Intuitively, if one firm does not do traditional advertising and relies on poaching, the competitor is forced to do traditional advertising, otherwise there will be no search volume.

Consider an alternative model in which the firms first decide how much to spend on traditional advertising. Then, given the information on traditional advertising, the firms decide how much to spend on sponsored search advertising on their own keyword and on
the competitor’s keyword. Theorem 6.1.7 in the appendix shows that each firm will split its budget between the keywords in sponsored search advertising proportional to the search volume for that keyword. Therefore, if Firm $i$ spends $T_i$ on the traditional channel, its profit will be

$$\Pi_i = \alpha T_i + \min \left( \left( \frac{B - T_i}{2B - T_i - T_j}, \frac{B - T_i}{R} \right), \right),$$

where $j = 3 - i$ represents the index of the other firm. The equilibrium profits of the firms with this new formulation remain the same as those in Section 2.3. Moreover, the poaching and non-poaching equilibria discussed in Section 2.3 are also equilibria of this game. To intuitively see why an asymmetric equilibrium will arise, if one firm spends zero on traditional advertising, its competitor’s best response is to use Traditional strategy. Given that the competitor uses Traditional strategy, spending zero on traditional channel and poaching on the competitor’s keyword is the best response. Also, similar to the analysis of Section 2.3, the best response to the Own strategy is to use the Own strategy.

#### 2.6.7 Consumers’ Purchase Model and Price Competition

In our basic model, we assumed that product prices are determined exogenously, and consumers purchase passively at the price offered to them by the firm whose traditional or sponsored advertisement they see most recently. In this extension, we model price competition between firms using a model in which consumers are horizontally differentiated. We assume that consumers get aware of a firm only if they see an ad of the firm, which can either be a traditional ad or an ad in sponsored search. Therefore, consumers that are poached become aware of both firms and compare prices across firms while making their purchase decisions, which leads to price competition. The consumers who see ads from only one firm do not compare prices as they are not aware of the second firm. More details of the model are available in Section 6.1.4 in the appendix.

We solve the model numerically and confirm that our original results are robust under price competition—equilibria exist in which one firm focuses on traditional advertising and the other focuses on poaching, and the search engine’s revenue is maximized with a medium level of penalty on poaching. A new interesting result from this model is that the poaching firm sets a lower price than the other firm. In this way, the poaching firm can maximize the effect of poaching on its competitor’s keyword and win more of the comparison shoppers. The firm that is being poached does not decrease the price as much because it is benefiting from the customers who are not aware of the product of the poaching firm (i.e., customers not influenced by sponsored search).

#### 2.7 Empirical and Anecdotal Support

The examples in the introduction section clearly show the existence of poaching in sponsored search advertising. We now provide empirical and anecdotal evidence to support some of our results. Note that the evidence we present is not a direct test of hypotheses from our model; it is only meant to offer supporting evidence for certain results. We leave more careful empirical tests for future work.
Short Term Poaching Behavior

We show that if a firm runs an effective traditional advertising campaign providing an impe-
tus to search activity for its keyword on search engines, competitors respond with increased poaching on this firm’s keyword in sponsored search advertising. To show this, we consider the time period before and after Super Bowl 2012. Advertising on TV during the Super Bowl is very salient and is known to reach millions of viewers. Which firms will advertise on Super Bowl in a particular year is known a few days in advance of the event. Based on the buzz in the popular press, we collected keywords related to the names of some companies and their specific products for which ads were expected to be shown during Super Bowl 2012. Based on this first list, we also collected keywords related to the names of the companies and products that are close competitors but were known to not be advertising during Super Bowl 2012. This gave us the following lists:

- Advertisers: Camry, Toyota, CR-V, Honda, Chrysler, GS 350, Lexus, Audi, Acura, Volt, Chevy, BMW (cars); Dannon (yogurt); Taxact (tax software); Go Daddy (WWW domain name registration); Etrade (online trading);
- Non-advertisers: Ford, Infiniti, Nissan, Mercedes Benz (cars); Yoplait (yogurt); Turbo-tax, H&R Block (tax software); Networkolutions (WWW domain name registration); TD Ameritrade, Scottrade, Fidelity (online trading).

Super Bowl 2012 was held on February 5, 2012. For each keyword we consider, for three days before and after the game was aired (using the local time of the start of the game to determine “before” and “after”) we collected data on its search volume at Google using Google Analytics (note that Google provides an index of search volume for every keyword, not the exact search volume). We also repeatedly queried Google with each of the above keywords, roughly once every hour, and crawled all the sponsored search results. Based on this, we monitor changes in keyword traffic and the degree to which each keyword is poached by competitors. We find that for the keywords in the advertised category, the average search volume was higher by 44.8% in the three days after Super Bowl as compared to the three days before Super Bowl (in agreement with Joo et al. (2011)), while for the keywords in the non-advertised category, the average search volume showed only a modest increase of 3.7%. In terms of poaching activity, poaching on keywords in the advertised category increased by 55.6% on average in the days after Super Bowl, while poaching on keywords in the non-advertised category decreased by 39.4% on average. We also find that, roughly between three days to a week after Super Bowl, search traffic and poaching activity return to pre-Super Bowl levels. This analysis shows that firms that run effective short-term traditional advertising campaigns leading to an increase in online keyword search activity are poached upon more by their competitors, supporting our results.

11Google is known to employ algorithms to modify the ads displayed in the sponsored search list when repeated requests come from the same IP address. To circumvent this issue, we queried using a randomized IP identity to prevent the search engine from identifying that repeated requests are coming from the same machine.
Long Term Poaching Behavior

We now show that, over a long time span, “weaker” firms in an industry poach more and are poached upon less, while “stronger” firms poach less and are poached upon more. We consider the following industries and firms:

- Insurance: Geico, Progressive, Allstate, Statefarm
- Online Trading: Etrade, Scottrade, TD Ameritrade, Sharebuilder
- E-Readers: Kindle, Nook, Playbook
- Pornography: Playboy, Penthouse, Hustler, Fling

To find the pattern of interest, we collect the following data. We take the names of each of the above companies as keywords searched on the search engine Google. For the time period from August 2010 to November 2011, we collect two types of data for each firm. First, using Google Insights, we collect monthly search traffic on Google for every company’s keyword. Second, using the website Spyfu.com, we collect monthly poaching data for every company’s keyword. Spyfu periodically queries Google with millions of keywords and crawls the sponsored search results. From these data, for each industry, we can reconstruct who poached on whom and how frequently, and construct monthly indices for the same. (Note that all the keywords of our interest were crawled by Spyfu.)

Using the monthly data for the sixteen months, we find the correlations (within industry) between (i) keyword traffic for a firm and the frequency with which this firm’s keyword was poached by other firms, and (ii) keyword traffic for a firm and the frequency with which it poached other firms’ keywords. The resulting numbers are provided in Table 2.4. The first row of the table shows that, for every industry, the correlation between keyword traffic and frequency of being poached is positive. The second row of the table shows that, for every industry, the correlation between keyword traffic and frequency of poaching others is negative. If we assume that higher keyword traffic is correlated with higher spend on awareness-generating advertising (as found by Joo et al. (2011) for television advertising), then the correlation numbers we find indicate that firms that spend more on awareness-generating advertising poach less, and vice versa, which supports our findings from the analytical model.

<table>
<thead>
<tr>
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<th>E-Readers</th>
<th>Pornography</th>
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<td>-0.22</td>
<td>-0.33</td>
<td>-0.66</td>
</tr>
</tbody>
</table>

Table 2.4: Correlation table for long-term poaching behavior

Search Engine Policy on Poaching

The policy that search engines follow regarding poaching is somewhat perplexing—they allow advertising by competitors on trademarked keywords (such as brand and company names), yet still employ keyword relevance measures to handicap poaching firms. Interestingly, our result in Proposition 2.5.1 suggests that this should exactly be the policy of the search

\[12\] We also consider popular alternative ways of spelling these names. For example, for the company Etrade, we use the keywords “Etrade,” “E-Trade” and “E Trade.”
engines because it gives them the flexibility to allow and make profit from poaching when beneficial to them (cases in which a smaller-budget firm has a large incentive to poach even if the search engine makes it more expensive to poach through the keyword-relevance penalty), and prevent poaching when it hurts their profit (cases in which firms’ budgets are comparable). Furthermore, we find that the weak firm practices poaching and benefits from it, the strong firm is hurt from poaching, and the search engine benefits from poaching by the weak firm. These results from the model support the observation that some leading firms in their respective industries (e.g., Rosetta Stone and Louis Vuitton) sued search engines in an effort to prevent them from following a policy of allowing bidding on trademarks by competitors (paidContent.org 2010, 2011). The search engines won these lawsuits and have continued to allow poaching on trademarked keywords; at the same time, they continue to use keyword relevance scores to handicap poaching. In the above examples, the predictions from our model are in close agreement with the actual behavior of the strong firms, the weak firms, and the search engines.

2.8 Discussion

In this research, we study the poaching behavior of firms in sponsored search advertising. A firm can spend on traditional channels of advertising such as television, print and radio to create awareness, attract customers, and increase the search volume of its keyword at a search engine. Alternatively, a firm may limit its awareness-creating activities and spend its budget on stealing the potential customers of its competitor by advertising on the competitor’s keyword in sponsored search, which we call poaching.

We find that even when firms are identical, they may follow different advertising strategies—one firm focuses on the traditional channel and spends a larger fraction of its budget for creating awareness, while the other firm spends a larger fraction of its budget on poaching. Surprisingly, the best response of the firm being poached is to accommodate poaching rather than retaliate. Although poaching seems to be beneficial for the search engine as it increases the competition on the keywords, we find that it actually may decrease the search engine’s revenue. Since poaching increases prices (bids) in sponsored search, thus increasing per-customer acquisition costs in this channel, the firms may spend less on sponsored search. Therefore, the search engine may increase its revenue by making poaching harder for the firms and keeping bids in check.

When the firms are asymmetric, and the advertising budget of one firm is significantly larger than the advertising budget of the other firm, there is an interesting twist in the above results. First, the stronger firm does not want to poach while the weaker firm has much more incentive to poach (as compared to the symmetric case). Furthermore, unlike the case of symmetric firms, poaching may increase the search engine’s revenue. Since the stronger firm has a large search volume, the effect of the weaker firm’s poaching is small. In other words, poaching of the weaker firm does not make sponsored search much less efficient for the stronger firm. Thus, the stronger firm keeps almost the same portion of its budget in sponsored search. On the other hand, the weaker firm does not need to create awareness and can spend its entire budget in sponsored search, which increases the search engine’s revenue.

We find that in the asymmetric firms’ case, the best strategy for the search engine is to
handicap poaching but not too much so that the weak firm still prefers to poach. This handicap can be implemented by charging the poaching firm a higher price than the non-poaching firm for the same keyword. Interestingly, we see that well-known search engines, e.g., Google, Yahoo! and Bing, have already implemented such penalties through “keyword relevance” multipliers. A firm has to pay higher price than its competitor for appearing in response to the competitor’s keyword, even if it has the same click-through-rate and quality measures as its competitor. By including keyword relevance measures in our model, we find that it may indeed be optimal for the search engine to use a medium level of penalty to maximize its revenue. Our results agree with the industry observations that the search engines, when sued by firms for allowing poaching, defended their practice of allowing bids on trademarked keywords, but are also penalizing poaching through keyword relevance multipliers.

We also consider various extensions of the model which confirm the robustness of our results and provide additional insights. Specifically, we consider an extension in which one firm has some search volume for its keyword even without recent awareness-generating advertising (say, because of previous reputation). We find that, surprisingly, the firm that has higher exogenous search volume due to reputation-based customer awareness has greater incentive to invest in traditional advertising to drive even more search volume to its keyword. In another extension, where the firms compete on price, we find that the poaching firm has incentive to set a lower price than its competitor.

Our work sheds light on the poaching behavior of firms in a multi-channel advertising setting. There are many other related problems that may be studied in future work. In particular, the firms are not vertically differentiated in our model. Perhaps a joint model of our work and Desai et al. (2011), that allows differentiation in a multi-channel advertising model, would be an interesting direction for future work. Another interesting direction to look at is the consequences of poaching among partners. For example, online travel agencies such as Orbitz bid on keywords such as “Sheraton Hotel in San Francisco” trying to steal and resell the potential customers of Sheraton back to Sheraton. This poaching not only decreases Sheraton’s profit from its own customers (because it has to share a part of the revenue with Orbitz for delivering this customer), but also increases the price of sponsored search advertising for Sheraton. It would be interesting to know how partners should react to such poaching behavior.

Credits: The results in this chapter are joint work with Kinshuk Jerath and Kannan Srinivasan.
Chapter 3

Budget Constraints in Sponsored Search Advertising

3.1 Introduction

In sponsored search auctions, in addition to a “per-click” value, the advertisers specify a daily/monthly maximum budget. Allocation and pricing is then determined by a complex algorithm which makes sure that the advertisers are not, per-click, charged more than their stated values and also are not charged more than their total budget in a day/month.

Advertisers true (estimated) values per-click and daily budgets are, of course, their private information and given any allocation and pricing rule they will act strategically in bidding their values and budgets. It is then natural to ask whether there is any mechanism in which the participants would prefer to truthfully reveal their types—per-click values and daily budgets. Then there will not be any “gaming” of the mechanism and socially efficient allocations can be implemented. Implementing a truthful mechanism not only improves efficiency; it also makes the game easier to play especially for less sophisticated advertisers. Finally, truthfulness (semi-truthfulness) could improve the stability of the auction: since truthful bidding is dominant strategy, advertisers do not have to respond to any change in strategies of the other advertisers.

Second-price auctions in single unit auction problems and different versions of Vickrey-Clark-Groves mechanisms in more general setups have been very successful in implementing socially efficient allocations in “dominant strategies.” Unfortunately, a recent impossibility result by Dobzinski et al. (2011) precludes the existence of a truthful mechanism with Pareto optimal allocations for multi-unit auctions with budget constraints.

The auction theory literature has paid relatively little attention to the case of budget-constrained bidders. However, budget constraints become quite relevant in multi-unit auctions especially when the auctioned items are of significantly high value for a single bidder to buy all units. Online advertisement auctions are arguably one such setup. Perhaps that is why the advertisers have to specify “a value per click” and “a daily maximum budget” for the advertisement auctions. The presence of budget constraints introduces important differences into traditional auction theory.
3.1.1 Related Literature

Prior literature on budget constraints in auctions have mainly focused on Bayesian Nash equilibria. The main motivation in this line of work was the constrained efficiency and/or revenue optimality in the presence of budget constraints.¹ Motivated by online advertisement auctions, we consider ex-post rather than Bayesian Nash equilibria, where the key efficiency criterion if that of Pareto optimality. The auctioneer’s revenue is also an important additional consideration in the online auction setting. In these settings, budgets are usually private information. In a recent paper, Dobzinski et al. (2011) prove an impossibility result in a setup where the budgets are private information. They first consider the case in which the budgets of all players are common knowledge, and show that the unique mechanism which is truthful and efficient is a variation of the Ausubel (2004) auction. Then by showing that the unique mechanism is not truthful if budgets are private information, they conclude that there exists no mechanism that is individually rational, truthful, and Pareto-optimal.²

3.1.2 Contribution

In this chapter, we model an environment where bidders have private constant marginal values and private budget constraints. Based on the above discussion, we are interested in designing a mechanism that has good Pareto optimality and revenue properties. As a starting point, consider selling a divisible item via a Vickrey auction when bidders have constant marginal values up to some limit (to keep the model simple, the limits are expressed not in terms of budgets, but as quantities). As an example, consider 6 bidders where bidders 1, 2, and 3 get positive marginal value up to 0.3 units and bidders 4, 5, and 6 get positive marginal value up to 0.4 units. The Vickrey auction distributes the objects according to the most efficient allocation and charges each winner the externality she imposes on the losing bidders. If bidders 1, 2, and 4 have the highest three marginal values and bidders 3 and 5 have the next two highest marginal values, in the Vickrey auction bidders 1, 2, and 4 will be awarded 0.3, 0.3, 0.4 units respectively. Moreover, bidders 1 and 2 will be paying bidder 3’s value per unit, and bidder 4 will be paying bidder 3’s value per unit up to 0.3 units, then would be paying bidder 5’s value for the remaining 0.1 units. This example assumes common knowledge of the limits, and if that is the case, the Vickrey auction allocates the objects efficiently in dominant strategies.

We propose a natural generalization of the Vickrey auction—Vickrey with Budgets—and show that it yields good revenue and Pareto optimality properties. As in the Vickrey

¹Two of the earliest papers regarding budget-constrained bidders are Che and Gale (1998) and Benoit and Krishna (2001). Che and Gale (1998) show that in the presence of budget constraints, first-price auctions may yield higher expected revenue than second-price auctions; Benoit and Krishna (2001) show that for multiple objects, the sequential auction may yield more revenue than the simultaneous ascending auction. Maskin (2000) considers the problem of finding the efficient auction subject to the buyers’ budget constraints. On the other hand, Laffont and Roberts (1996), Malakhov and Vohra (2008), and Pai and Vohra (2010) consider the problem of optimal auctions when bidders have budget constraints.

²There is a quite large literature in the computer science community that tries to approximate optimal revenue in the presence of budget constraints. For instance, see Borgs et al. (2005), Abrams (2006), Ashlagi et al. (2010), Bhattacharya et al. (2010a), Feldman et al. (2008).
auction, we prove that understating budgets or values is weakly dominated in our mechanism. However, unlike Vickrey, overstating budget or value might be beneficial to the bidders. Nevertheless, we show that such deviations lead to higher revenue for the auctioneer than truthful revelations of values and budgets.

The idea of Vickrey with Budgets is the very idea of the Vickrey auction mentioned above. Taking budget constraints into account, we charge the winners, per item, the value of the highest-value loser, but only up to this loser’s budget. After the highest-value loser’s budget is exhausted, we start charging the winners the second-highest loser’s value, up to her budget and so on. Given this pricing idea, the winners and losers are determined via a cut-point to clear the market, i.e. all the available units are sold. The cut-point may lead to the lowest-value winner being partially allocated, while the other bidders are all full winners or full losers.

Vickrey with Budgets has a number of desirable properties. First of all, bidders can only benefit by overstating their values or budgets, a deviation that is the most desirable for the auctioneer. Secondly, the allocation in the equilibrium of Vickrey with Budgets is nearly Pareto optimal in the sense that all full winners’ values are above all full losers’ values. Thirdly, Vickrey with Budgets reduces to a second-price auction when there are no budget constraints.

3.2 The Model and Vickrey with Budgets

There is a single unit of a divisible good for sale. There are \( n \) bidders with a linear demand up to their budget limits. Specifically, each bidder \( i \in N = \{1, \ldots, n\} \) has a two-dimensional type \((b_i, v_i)\) where \(b_i\) denotes her budget limit and \(v_i\) denotes her private value. Bidder \( i \)'s utility by getting \( q \) fractional units of the good and paying \( p \) is given by

\[
u_i(q, p) = \begin{cases} qv_i - p & \text{if } p \leq b_i \\ -C & \text{if } p > b_i \end{cases}
\]

where \( \infty \geq C > 0 \).

We are interested in mechanisms with good incentive, efficiency, and revenue properties to sell this good to \( n \) bidders. The equilibrium concept we use is that of an \textit{ex-post Nash equilibrium}. In an ex-post Nash equilibrium, no bidder wants to deviate after she observes all other players’ strategies.

We focus on direct mechanisms in which bidders announce their types (values and budgets). A mechanism consists of an allocation rule (how many units to allocate to each bidder) and a pricing rule (how much to charge each bidder). It takes the announcements as inputs and produces an allocation and a pricing scheme as the output. We consider pricing rules

\footnote{There is a caveat here, which is that the lowest-value winner might not be able to exhaust all of her budget. Then all higher-value bidders are charged first at the lowest-value winner’s value up to her unused budget. The pricing for the lowest-value winner starts from the highest-value loser.}

\footnote{\( C \) can also be a function of \( p \). Our results are not affected as long as bidders get a negative utility once they exceed their budgets.}
such that bidders who are not allocated any units (losers) do not pay anything. Bidders who are allocated nonzero units (winners), however, will be charged a positive price. Let us first introduce a general abstract pricing rule.

**Definition** The price is set according to a pricing function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \), if the marginal price of the next unit is \( \alpha (y) \) dollars for a buyer who has already spent \( y \) dollars in the market. In other words, if the pricing of an item is set according to \( \alpha \), a buyer with \( b \) dollars can afford

\[
x (\alpha, b) = \int_0^b \frac{1}{\alpha (y)} dy
\]

units of the item. We are interested in pricing rules \( \alpha (\cdot) \) that are nonincreasing and positive. Hence, we assume \( \alpha (y) \leq \alpha (y') \) for all \( y \geq y' \) and also \( \alpha (y) > 0 \) for all \( y \).

The following definition is also convenient for later discussions.

**Definition** (Shifted pricing) For a given pricing function \( \alpha \) and a positive real number \( z \), we define the pricing function \( \alpha^z (y) \) as:

\[
\alpha^z (y) = \alpha (z + y).
\]

Less formally, \( \alpha^z (y) \) is the pricing function obtained by shifting \( \alpha (y) \), \( z \) units to the right. Note that we have, for any \( z \in [0, b] \),

\[
x (\alpha, b) = x (\alpha, z) + x (\alpha^z, b - z).
\]

Throughout the proofs of our results, we sometimes make use of the terms “better (or worse) pricing function” and “getting to lower prices.” We say that \( \alpha \) is a better pricing function than \( \alpha' \) for a bidder if \( \alpha (y) \leq \alpha' (y) \) for all \( y \). We say that \( \alpha \) gets to lower prices than \( \alpha' \) for a bidder with budget \( b \) if marginal payment at \( b \) is lower with \( \alpha \) than with \( \alpha' \).

We are now ready to introduce our mechanism, which we name Vickrey with Budgets.

**Definition** A \( c \)-procedure cut algorithm takes budgets and values of the bidders \( (b, v) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \), a pricing rule \( \alpha (\cdot) \), and a real number \( c \in [0, \sum_{i=1}^n b_i] \) as input. First, it sorts bid and value vectors \( (b, v) \) in nonascending order of values and reindexes them so that \( v_1 \geq v_2 \geq \ldots \geq v_n \).

Then, it picks \( j \) such that \( c \leq \sum_{i=1}^j b_i \) and \( c > \sum_{i=1}^{j-1} b_i \). Let \( s = \sum_{i=1}^j b_i - c \). Procedure Cut sets the pricing function of bidders \( 1, \ldots, j - 1 \) to \( \alpha^c \) and the pricing function of bidder \( j \) to \( \alpha^{c+s} \), where \( \alpha (\cdot) \) is a step function defined by (reindexed) \( (b', v') \): \( \alpha (y) = v_i \) for \( y \in (\sum_{k=1}^{i-1} b_k, \sum_{k=1}^i b_k] \). The allocation of each bidder \( 1, \ldots, j - 1 \) is such that she spends all her budget, i.e. \( x_i = x (\alpha^c, b_i) \) for \( i = 1, \ldots, j - 1 \). The allocation of bidder \( j \) is such that she spends \( b_j - s \) of her budget, i.e. \( x_j = x (\alpha^{c+s}, b_j - s) \). Bidder \( j \)'s unused budget is denoted by \( s \), where \( s \in [0, b_j] \). All bidders \( j + 1, \ldots, n \) get no allocation and pay nothing.

\footnote{It breaks ties among equal valued bidders arbitrarily.}

\footnote{Note that after reindexing, budgets are not necessarily sorted in a descending way. A bidder with a high valuation could have a small budget.}
Define $X(c, (b, v))$ to be the total number of units allocated to all bidders, i.e. $X(c, (b, v)) = \sum_{i=1}^{j} x_i$. We show in Proposition 1 below that there is a unique $c^*$ that satisfies $X(c^*, (b, v)) = 1$. 

Vickrey with Budgets is defined to be the $c^*$-procedure cut algorithm. Bidders 1, ..., $j$ are called full winners, bidder $j$ is called a partial winner (or a cut-point bidder), and bidders $j + 1$, ..., $n$ are called losers.

In other words, Vickrey with Budgets takes the vectors $(b, v)$ and sorts them in nonascending order of values, calculates the unique cut-point $c^*$ which sells one unit according to the following pricing function: Each full winner (bidders 1, ..., $j - 1$) pays $v_j$ per unit up to a budget of $s$, then pays $v_{j+1}$ per unit up to a budget of $b_{j+1}$, then pays $v_{j+2}$ per unit up to a budget of $b_{j+2}$, and so on, until their budgets are exhausted; the partial winner (bidder $j$) pays $v_{j+1}$ per unit up to a budget of $b_{j+1}$, then pays $v_{j+2}$ per unit up to a budget of $b_{j+2}$, and so on, until she spends $b_j - s$; full losers pay nothing.

**Proposition 3.2.1.** $X(c, (b, v))$ is strictly increasing and continuous in $c$. Therefore, there is a unique $c^*$ that satisfies $X(c^*, (b, v)) = 1$

**Proof.** In the Appendix.

Let us the denote the revenue of Vickrey with Budgets by $R^B(b, v)$. Note that $R^B(b, v) = c^*$ where $X(c^*, (b, v)) = 1$.

**Proposition 3.2.2.** $R^B(b, v)$ is nondecreasing in $b$ and $v$.

**Proof.** In the Appendix.

### 3.3 Incentives, Revenue, and Near Pareto Optimality

#### 3.3.1 Incentives

In this section, we show that Vickrey with Budgets has good incentive properties. Specifically, we show that no bidder benefits from understating her value or budget. First we argue that three deviations that understate value or budget or both are weakly dominated in ex-post equilibria. Then we consider two other deviations that might potentially decrease revenue and argue that either they are unreasonable or they result in higher revenue.

**Proposition 3.3.1.** For any bidder $i$ with types $(b_i, v_i)$, bidding $(b_i, v_i^-)$ weakly dominates bidding $(b_i, v_i)$ for $v_i^- < v_i$.

**Proof.** Consider any $(b_{-i}, v_{-i})$. First of all, if $i$ becomes a loser by bidding $(b_i, v_i^-)$, her utility cannot increase with this deviation. This is because losers’ utilities are zero, and by construction, a bidder with type $(b_i, v_i)$ achieves a nonnegative utility by bidding $(b_i, v_i)$. We will look at the possible cases one by one.

- If $i$ loses by bidding $(b_i, v_i)$, then she will lose by bidding $(b_i, v_i^-)$ (since the pricing function gets better for the winners). Hence her utility cannot increase by this deviation.
• If \( i \) is a partial winner by bidding \((b_i, v_i)\) and by bidding \((b_i, v_i^-)\) she is still a partial winner, then she will have the same pricing function but she will be able to use less of her budget (since the pricing function for full winners becomes better); hence her utility cannot increase. Bidder \( i \) cannot become a full winner by bidding \((b_i, v_i^-)\) when she is a partial winner by bidding \((b_i, v_i)\).

• If \( i \) wins by bidding \((b_i, v_i)\) and by bidding \((b_i, v_i^-)\) she is still a winner, her utility does not change. This is because Vickrey with Budgets ignores the value of winners in the pricing calculation. If \( i \) wins by bidding \((b_i, v_i)\) and bidding \((b_i, v_i^-)\) makes her a partial winner, then the original partial winner \( j \) (with an unused budget \( s \)) has to be a winner after the deviation. We argue that \( i \)'s utility decreases. It is true that \( i \) would get the items at a lower per-unit price after the deviation, but at the same time she is using less of her budget. The argument is that, by this deviation, \( i \) cannot get to lower-priced items, and this follows from the fact that revenue of Vickrey with Budgets cannot decrease after the deviation. More formally, let us denote the unused budget of \( i \) after the deviation by \( s' \). We know that \( s' \geq s \) (because revenue cannot increase). Bidder \( i \)'s utility difference with the deviation can be shown to be nonpositive (where \( \alpha \) and \( c \) are defined with respect to \((b, v)\)) as follows.

\[
\begin{align*}
(x(\alpha^{c+s}, b_i - s') v_i - (b_i - s')) - (x(\alpha^c, b_i) v_i - b_i) \\
= (x(\alpha^{c+s}, b_i - s') - x(\alpha^c, b_i)) v_i + s' \\
\leq (x(\alpha^{c+s'}, b_i - s') - x(\alpha^c, b_i)) v_i + s' \\
= (x(\alpha^{c+s'}, b_i - s') - (x(\alpha^c, s') + x(\alpha^{c+s'}, b_i - s'))) v_i + s' \\
= s' - x(\alpha^c, s') v_i \\
\leq s' - \frac{s'}{v_i} v_i \\
= 0
\end{align*}
\]

where the first inequality follows from \( s' \geq s \) and the second inequality follows from \( \alpha^c(y) \leq v_i \).

\[\square\]

Proposition 3.3.2. For any bidder \( i \) with type \((b_i, v_i)\), bidding \((b_i, v_i)\) weakly dominates bidding \((b_i^-, v_i)\) for \( b_i^- < b_i \).

Proof. Consider any \((b_{-i}, v_{-i})\): First of all, as in the previous proof, if \( i \) becomes a loser by bidding \((b_i^-, v_i)\), her utility cannot increase with this deviation. We look at the possible cases one by one.

• If \( i \) loses by bidding \((b_i, v_i)\), then she will lose by bidding \((b_i^-, v_i)\) (since the pricing function gets better for the winners).

• If \( i \) is a partial winner by bidding \((b_i, v_i)\) and by bidding \((b_i^-, v_i)\) she is still a partial winner, then she will have the same pricing function but will be able to use less of
her budget (since the pricing function for winners becomes better); hence her utility cannot increase. Bidder \( i \) cannot become a full winner by bidding \((b_i, v_i^-)\) when she is a partial winner by bidding \((b_i, v_i)\).

- If \( i \) wins by bidding \((b_i, v_i)\) and bidding \((b_i^-, v_i)\) makes her a partial winner, then \( i \) would be worse off with this deviation. This is because she is using less of her budget, and her pricing got worse. If \( i \) is a full winner by bidding \((b_i, v_i)\) and bidding \((b_i^-, v_i)\) leaves her as a full winner, we can argue that her utility decreases. It is true that \( i \) may get the items at a lower per-unit price after the deviation, but at the same time she is using less of her budget. The argument is that by this deviation \( i \) cannot get to lower-priced items, which follows from the fact that the revenue of Vickrey with Budgets cannot increase after the deviation. More formally, bidder \( i \)'s utility difference with the deviation can be shown to be nonpositive as follows. Here \( \alpha \) and \( c \) are defined with respect to \((b, v)\) and \( c' (\leq c) \) is the revenue of Vickrey with Budgets after the deviation.

\[
\begin{align*}
\left( x \left( \alpha^{c+b_i-b_i^-}, b_i^- \right) v_i - b_i^- \right) - \left( x \left( \alpha^c, b_i \right) v_i - b_i \right) \\
= \left( x \left( \alpha^{c+b_i-b_i^-}, b_i^- \right) - x \left( \alpha^c, b_i \right) \right) v_i + b_i - b_i^- \\
\leq \left( x \left( \alpha^{c+b_i-b_i^-}, b_i^- \right) - x \left( \alpha^c, b_i \right) \right) v_i + b_i - b_i^- \\
= \left( x \left( \alpha^{c+b_i-b_i^-}, b_i^- \right) - x \left( \alpha^c, b_i - b_i^- \right) + x \left( \alpha^{c+b_i-b_i^-}, b_i^- \right) \right) v_i + b_i - b_i^- \\
= b_i - b_i^- - x \left( \alpha^c, b_i - b_i^- \right) v_i \\
\leq b_i - b_i^- - \frac{b_i - b_i^-}{v_i} v_i \\
= 0
\end{align*}
\]

where the first inequality follows from \( c \geq c' \) and the second inequality follows from \( \alpha^c(y) \leq v_i \) for all \( y \).

Now we can argue that bidding \((b_i^-, v_i^-)\) for \( b_i^- < b_i \) and \( v_i' < v_i^- \) is weakly dominated by bidding \((b_i, v_i)\). This follows from the proofs above. The two previous propositions imply that both \((b_i^-, v_i)\) and \((b_i, v_i^-)\) dominate \((b_i^-, v_i^-)\) when \( b_i^- < b_i \) and \( v_i^- < v_i \). Applying either of them one more time, we have the following result.

**Proposition 3.3.3.** For any bidder \( i \) with type \((b_i, v_i)\), bidding \((b_i, v_i)\) weakly dominates bidding \((b_i^-, v_i^-)\) for \( b_i^- < b_i \) and \( v_i^- < v_i \).

Propositions 3.3.1, 3.3.2 and 3.3.3 establish that these revenue-decreasing deviations should not occur in equilibrium (they are weakly dominated). There are two deviations, however, that may increase or decrease the revenue. These deviations are “underestimating

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7To see why bidder \( i \)'s pricing function after the deviation is \( \alpha^{c+b_i-b_i^-} \), note that her pricing function is \( \alpha^c \) according to types after the deviation, and this translates to \( \alpha^{c+b_i-b_i^-} \) with the original types.
budget and overstating value” and “overstating budget and understating value.” We now show that the former deviation is not reasonable in the sense that it could be a best response only when the utility with that strategy is zero. Then we show that the latter deviation could happen in an equilibrium, yet whenever it is a (strict) profitable deviation from truthful revelation, the revenue increases with the deviation.

**Proposition 3.3.4.** For any bidder $i$ with type $(b_i, v_i)$, for $b_i^- < b_i$ and $v_i^+ > v_i$, bidding $(b_i^-, v_i^+)$ can never be in the set of best responses unless bidder $i$’s utility in her best response is 0.

**Proof.** Given $(b_{-i}, v_{-i})$, suppose that $(b_i^-, v_i^+)$ is a best response for $i$ where $b_i^- < b_i$ and $v_i^+ > v_i$. Since bidding $(b_i, v_i)$ would give nonnegative utility to bidder $i$, the utility by bidding $(b_i^-, v_i^+)$ has to be nonnegative. We claim that bidding $(b_i, v_i^+)$ is a better response than $(b_i^-, v_i^+)$, and it is strictly better when the utility by bidding $(b_i, v_i^+)$ is strictly positive. This implies $(b_i^-, v_i^+)$ could be a best response only when bidder $i$’s utility in her best response is 0.

Suppose bidder $i$’s utility by bidding $(b_i^-, v_i^+)$ is nonnegative, and consider the utility difference between bidding $(b_i^-, v_i^+)$ and bidding $(b_i, v_i^+)$. The utility difference is clearly zero if $i$ is a loser in both cases. For all other cases, $i$ would be either a partial winner or a full winner by bidding $(b_i, v_i^+)$. Then we can see that bidding $(b_i, v_i^+)$ gives a higher utility than bidding $(b_i^-, v_i^+)$. The argument is the same as in the proof for Proposition 3.3.2 by bidding an extra budget of $b_i - b_i^-$, bidder $i$ can get extra items at a per-unit price lower than her value, leading to a nonzero increase in her utility.

In other words, we should not expect to see $(b_i^-, v_i^+)$ to be reported in an ex-post equilibrium, since it is worse than either $(b_i, v_i)$ or $(b_i, v_i^+)$. 

**Proposition 3.3.5.** For any bidder $i$ with type $(b_i, v_i)$, for $b_i^+ > b_i$ and $v_i^- < v_i$, whenever bidding $(b_i^+, v_i^-)$ brings a higher utility to $i$ than bidding $(b_i, v_i)$, the auctioneer’s revenue with $(b_i^+, v_i^-)$ is not lower than the revenue with bidder $i$ bidding $(b_i, v_i)$.

**Proof.** Given $(b_{-i}, v_{-i})$, for some $b_i^+ > b_i$ and $v_i^- < v_i$, suppose that $u_i((b_{-i}, b_i^+), (v_{-i}, v_i^-)) > u_i((b_{-i}, b_i), (v_{-i}, v_i))$. Since bidder $i$ is budget constrained, she will have to be a partial winner by bidding $(b_i^+, v_i^-)$ (if she is a full winner her utility would be $-C$, and if she is a loser her utility would be 0).

- If she loses by bidding $(b_i, v_i)$, the auctioneer’s revenue clearly increases with $(b_i^+, v_i^-)$. This is because $i$’s ranking with $v_i^-$ is not higher than that with $v_i$, and so by deviating from $(b_i, v_i)$ to $(b_i^+, v_i^-)$, all full winners remain full winners and $i$ becomes a partial winner.

- If she is a full winner by bidding $(b_i, v_i)$, the partial winner with $(b_i, v_i)$ has to become a full winner after $i$ deviates to $(b_i^+, v_i^-)$. Otherwise, $i$ would be worse off by bidding $(b_i^+, v_i^-)$ as she will have a worse pricing function. In this case the revenue has to increase. The argument is that, for this deviation to be beneficial, $i$ has to get lower priced items after the deviation. For this to be the case, the partial winner’s unused budget before the deviation plus $i$’s used budget after the deviation, has to be greater.
than \( i \)'s budget \( b_i \). But in this case, the revenue increases, since the new cut point is greater than the old one.

- If she is a partial winner by bidding \((b_i, v_i)\), we need to analyze two cases: (i) \( i \)'s ranking among the bidders is the same, or (ii) \( i \)'s ranking is different. For (i), the pricing for \((b_i, v_i)\) and \((b^+_i, v^-_i)\) are the same. Since bidder \( i \)'s utility by bidding \((b^+_i, v^-_i)\) is more than that by bidding \((b_i, v_i)\), this means \( i \) is using more of her budget with \((b^+_i, v^-_i)\). Therefore the revenue increases. For (ii), \( i \)'s ranking has to be worse with \((b^+_i, v^-_i)\).

Now, similar to the previous case, we argue that total budget of “new full winners” after the deviation plus the used budget of \( i \) after deviation has to be greater than \( b_i \). If that is not the case, \( i \) cannot get to lower prices.

In the above propositions we argued that playing \((b^-_i, v_i)\), \((b_i, v^-_i)\), or \((b^-_i, v^-_i)\) is not reasonable (they are dominated by \((b_i, v_i)\)); playing \((b^-_i, v^+_i)\) is not reasonable in a weaker sense (it is dominated by a combination of \((b_i, v_i)\) and \((b_i, v^+_i)\)); also, playing \((b^+_i, v^-_i)\) is reasonable only when it is done by a winner, who becomes a partial winner after deviation and increases the overall revenue. We call an equilibrium in which the strategies satisfy these conditions a refined equilibrium.

**Definition** A refined equilibrium is an equilibrium of Vickrey with Budgets where for all bidders \( i \), bidder \( i \) does not play \((b^-_i, v_i)\), \((b_i, v^-_i)\), \((b^-_i, v^-_i)\), or \((b^-_i, v^+_i)\). Moreover, a bidder \( i \) plays \((b^+_i, v^-_i)\) only when \( u_i((b^-_i, b^+_i), (v^-_i, v^-_i)) > u_i((b^-_i, b^-_i), (v^-_i, v^-_i)) \).

In other words, in a refined equilibrium, bidders never underestimate their budgets, and they underestimate their values only when they also simultaneously overstate their budgets, making them strictly better off than their truthful announcements. Recall that when \( u_i((b^-_i, b^+_i), (v^-_i, v^-_i)) > u_i((b^-_i, b^-_i), (v^-_i, v^-_i)) \), bidding \((b^+_i, v^-_i)\) makes \( i \) a partial winner after the deviation and the revenue is higher with \((b^+_i, v^-_i)\) than with \((b_i, v_i)\).

### 3.3.2 Revenue

There are eight possible kinds of deviations from the truthful revelation \((b_i, v_i)\). Five of them are discussed in the definition of a refined equilibrium. The remaining three of them, namely \((b_i, v^+_i)\), \((b^+_i, v_i)\), and \((b^+_i, v^+_i)\), can only increase the revenue by Proposition 3.2.2. Hence we have the following result.

**Theorem 3.3.6.** In a refined equilibrium of Vickrey with Budgets, revenue is bounded below by the revenue of Vickrey with Budgets with truthful revelations.

**Proof.** Consider any refined equilibrium of Vickrey with Budgets. Let \( b^-_i \) and \( v^-_i \) denote understating the types, and \( b^+_i \) and \( v^+_i \) denote overstating the types (with respect to true types). We know that \((b^-_i, v^-_i)\), \((b^-_i, v^-_i)\), \((b^-_i, v^-_i)\), and \((b^-_i, v^+_i)\) do not occur. Additionally, \((b^+_i, v^-_i)\) could occur only for the current cut-point bidder, and by Proposition 3.3.5, if we change it back to \((b_i, v_i)\), revenue cannot increase. Finally, the rest of the bidders are either bidding truthfully or using \((b_i, v^+_i)\), \((b^+_i, v_i)\), or \((b^+_i, v^+_i)\). In any case, changing their bid to
their truthful values cannot increase the revenue. Therefore, revenue in a refined equilibrium of Vickrey with Budgets is not smaller than revenue of Vickrey with Budgets with truthful revelations.

3.3.3 Near Pareto Optimality

We say that an allocation is Pareto optimal if there is no other allocation in which all players (including the auctioneer) are better off and at least one player is strictly better off.

In this setup, Dobzinski et al. (2008) has shown that Pareto optimality is equivalent to a “no trade” condition: an allocation is Pareto efficient if (a) all units are sold and (b) a player get a non-zero allocation only if all higher-value players exhaust their budgets. In other words, an allocation is Pareto optimal when, given the true value of the partial winner, winners and losers are ordered in the right way: all winners have higher values and all losers have lower values.

The following shows that in any ex-post Nash equilibrium of Vickrey with Budgets, the full winners and losers are ordered in the right way given the announced value of the partial winner.

**Theorem 3.3.7.** Consider any ex-post Nash equilibrium of Vickrey with Budgets where \( v_j \) is the announced value of the partial winner \( j \). Every bidder \( i \neq j \) who has a true value \( v^T_i > v_j \) is a full winner, and every bidder \( i \neq j \) who has a true value \( v^T_i < v_j \) is a loser in this equilibrium of Vickrey with Budgets.

**Proof.** First, consider a bidder \( i \) whose value is \( v^T_i > v_j \). We prove that she must be a full winner in equilibrium. Assume for the sake of contradiction that bidder \( i \) is a loser, so her utility is zero. If she deviates and bids \( v_j + \varepsilon \) (for \( 0 < \varepsilon < v^T_i - v_j \)) and her true budget, she will become either a full winner or the cut-point bidder (otherwise revenue of Vickrey with Budgets will decrease with this deviation, which is not possible because of Proposition 3.2.2). Obviously her utility becomes strictly positive with this deviation (her price per unit is at most \( v_j \)). We thus reach the necessary contradiction to her individual rationality.

Now consider a bidder \( i \) whose value is \( v^T_i < v_j \). Assume for the sake of contradiction that bidder \( i \) is a full winner. If \( b_i > s \), then she gets all items at a per-unit price \( v_j \), and hence she obtains a negative utility. If this is the case, she would be better off announcing her true valuations to guarantee a nonnegative payoff. If \( b_i < s \), then we argue that \( i \) would be better of by deviating to \((v_j - \varepsilon, b_i)\) for small enough \( \varepsilon \). Let us first look at the limiting case in which \( i \) deviates to \((v_j, b_i)\) and becomes the cut-point bidder. After this deviation, the unused budget of \( i \) would be exactly \( s \). The allocation of original full winners will not change; bidder \( j \) will be getting \( \frac{s}{v_j} \) more items by paying \( s \) more and bidder \( i \) will be getting \( \frac{s}{v_i} \) less items by paying \( s \) less. Therefore, bidder \( i \)'s utility increases by \( \frac{s}{v_i} (v_j - v^T_i) > 0 \) (in a sense by this deviation, bidder \( i \) is selling \( \frac{s}{v_j} \) units of the items to bidder \( j \) at the per-unit price of \( v_j \)). By deviating to \((v_j - \varepsilon, b_i)\), the

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\[ ^8 \] Maximizing social welfare dictates all items to be allocated to the bidder with the highest value, even if this bidder has a very small budget. We follow Dobzinski et al. (2008) and consider Pareto optimality as the appropriate efficiency concept.
original full winners’ allocations would slightly increase; therefore bidder $i$’s utility increase will be slightly smaller than \( \frac{\alpha}{v_j} (v_j - u_i^T) \). But for small enough \( \varepsilon \), it will always be positive, leading again to a contradiction.

This theorem establishes that given equilibrium cut-point value, all winners and losers will be correctly placed. But since the cut-point bidder may be misplaced, this does not imply full Pareto optimality. Consider the following example.

**Example** There are 2 units of the item to be sold, and there are four bidders with budget-value pairs \((18, 19)\), \((1, 9)\), \((7, 8)\), and \((10, 1)\). For this setup, it can be confirmed that bidders announcing their types (budget, value) as \((18, 19)\), \((1, 9)\), \((36, 18)\), and \((10, 1)\) constitute an ex-post equilibrium of Vickrey with Budgets. In this equilibrium, bidder 3 overstates her value and budget and becomes the partial winner. Although the full winners and the losers are rightly ranked according to the announced value of the partial winner, the allocation is not Pareto optimal. Bidder 3 gets a positive allocation even though bidder 2 has a higher value and zero allocation.

As an aside, note that the revenue of Vickrey with Budgets in this ex-post equilibrium is 18 while the revenue with truthful types is \( 18 + \frac{17}{9} \).

### 3.4 Discussion

In this essay, we have introduced a mechanism, Vickrey with Budgets, to sell a divisible good to a set of bidders with budget constraints. In this important setting, where a mechanism that is simultaneously truthful and Pareto optimal is precluded, our mechanism achieves good incentive, revenue, and efficiency properties. Specifically, in Vickrey with Budgets, (i) there are profitable deviations from truthful revelations of types, but these can only happen in a revenue-increasing way; and (ii) the equilibrium allocation is *nearly Pareto efficient* in the sense that full winners and losers are ordered in the right way given the announced value of the partial winner.

There are many ways our work can be generalized. In the context of online advertisement auctions, our model can be interpreted as “there is a *single* sponsored link that gets (normalized) 1 click a day and there are *n* advertisers.” However, in reality, there are many sponsored links. In generalized second-price auctions, studied by Edelman et al. (2007), the winner of the best item (first sponsored link) is charged the bid of the second-best item, the winner of the second-best item is charged the bid of the third-best item, and so on. In this environment there are no budget constraints and the second-highest bid is always the competitor of the highest value. The idea of Vickrey with Budgets can be applied in this setup with budget constraints. More specifically, it would be interesting to consider a model in which there are budget-constrained bidders and multiple slots available for a query in which an advertiser cannot appear in more than one slot per query.

\(^{9}\)There is an implicit continuity assumption here. However, it is not difficult to show that utilities of the bidders are continuous in type announcements.
In our model, we consider a setting of hard budget constraints in which the bidders cannot spend more than their budgets. Extending our results to a soft-budget problem in which bidders are able to finance further budgets at some cost is a promising direction.\footnote{See Gavious, Moldovanu and Sela (2002) where the utility of an agent is declining in price via “bid costs.”} One can model this kind of soft-budget constraint as specifying marginal value up to some budget, then specifying a smaller marginal value up to some other extra budget, and so on in a piecewise linear fashion. By replicating a bidder into as many copies as the number of pieces in her value/budget function, and allowing them all to participate in our mechanism, it seems reasonable that we may preserve some of the desirable properties of Vickrey with Budgets.

**Credits:** The results in this chapter are joint work with Isa Hafalir and R. Ravi.
Chapter 4

Exclusive Display in Sponsored Search Advertising

4.1 Introduction

As discussed in Chapter 1, the spectacular rise of sponsored search advertising in the last decade has been accompanied by constant effort from the search engines to refine their pricing mechanisms by gradually fixing the deficiencies in them, including developing new auction mechanisms to rank advertisers. Moreover, if one search engine introduces a profitable innovation, other search engines follow suit in a short time. As the industry matures, search engines are looking to further expand their bidding mechanisms by allowing advertisers to be more specific about their utilities and to express a richer set of preferences. For example, Google has been considering “hybrid” advertising auctions which allow advertisers to bid on a per-impression or a per-click basis for the same advertising space. Zhu and Wilbur (2011) show that such auctions can enhance both search engine revenue and the efficiency of advertisers’ allocation to positions.

A recent and very interesting development in this context has been the exploration by search engines of auction mechanisms that allow advertisers to bid for exclusive display in response to a user search. In other words, advertisers can bid for their ad to be the only ad displayed, rather than being one of many ads displayed. Exclusive display may be an attractive option for advertisers as an advertiser can create strong brand associations by being the only one displayed in response to certain keywords. For example, if the ad of only the manufacturer Olympus gets displayed in response to the keyword “digital camera,” it can be a significant branding advantage for Olympus over its competitors such as Canon and Nikon. Moreover, multiple ads shown next to each other may impose negative externalities on each other. For example, if a user who has searched for the keyword “car rental” clicks on the ad of Hertz, chances are that she will also go back to check the ads of some other companies displayed in the sponsored list, such as Avis and Budget, before finalizing the transaction with Hertz. These negative externalities, which can decrease the values of clicks to advertisers, will be smaller if only one ad is shown to the user.

Similar to the above, Desai et al. (2010) argue that when advertisers are listed next to each other, “context effects” influence users’ perceptions of their relative qualities, which
in certain cases can hurt the advertisers, especially the high-quality ones. Such effects may motivate advertisers to prefer exclusive display. In addition, showing only one ad might have positive effects on the users’ perception of the ads; if only one ad is shown, it is more likely that the user considers the ad more seriously. Therefore, it might change the concept of sponsored ads from “something annoying” to a “serious option” for some users. In summary, exclusive display will not only increase the expected clicks on an ad if the relevant keyword is searched, but may also increase the valuation per click for the advertiser. Therefore, advertisers may be willing to pay a higher price per click for having exclusive appearance for some keywords.

Interestingly, all three of the most popular search engines in the USA, namely Google, Yahoo! and Bing, have explored exclusive display auctions as part of their research efforts in the recent past. Google has considered displaying exclusive ads in response to user queries as part of its “perfect ad” initiative (Metz 2011). Yahoo! has, in fact, advanced even further and, in March 2011, has been granted patents on certain aspects of the idea of exclusive display and on two particular exclusive display auctions that it developed (U.S. Patent 20110071908 and U.S. Patent 20110071909). Our discussions with researchers and executives at Bing indicate that Microsoft is also exploring exclusive display auctions. These are strong indicators that exclusive display of ads may be adopted by popular search engines in the near future.

There is, however, also a debate about the value to search engines of switching to exclusive display mechanisms. For instance, Metz (2011) reports that high-level executives at Google were constantly going back and forth between adopting and not adopting exclusive display advertising. On the one hand, advertisers could be expected to pay more for exclusive display. On the other hand, many advertisers would not be displayed and Google would lose revenue from these advertisers. Eventually, due to a lack of proper understanding, Google chose to temporarily stay with the status quo of displaying multiple ads. Faced by the same questions, Yahoo! and Bing are conducting independent research on the value of exclusive display auctions.

Exclusive display of sponsored ads raises several questions. Is it advantageous for the search engine to switch to exclusive display, and why (or why not)? Should firms switch completely to exclusive display or should they adopt a “hybrid” format in which firms can bid for both multiple display and exclusive display, and the final outcome is decided by the search engine based on the submitted bids? What are the forces driving bidding behavior in exclusive display auctions? If firms can extend the current GSP mechanism in different ways to allow firms to bid for exclusive display, which of these auction mechanisms will provide the most revenue and under what conditions? Finally, what is the impact on the advertisers and on social welfare? In this essay, we take a step forward towards answering these questions.

Given that GSP is the auction mechanism used widely, we consider auction mechanisms that are extensions of GSP. We start by assuming that each advertiser can have different per-click valuations for clicks obtained when it is displayed with other advertisers (multiple display) and clicks obtained when it is the only one displayed (exclusive display). We analyze two auction mechanisms that are conceptually simple extensions of GSP and were recently patented by Yahoo! as the main candidates for implementation. In these auctions, each advertiser submits a two-dimensional bid—its maximum willingness to pay per click for multiple display, and its maximum willingness to pay per click for exclusive display. In
the first auction, \( NP_{2D} \), the next-price rule of \( GSP \) is extended to two dimensions—every advertiser has to pay the minimum amount necessary to maintain the outcome configuration (multiple or exclusive display) and its position within the configuration. In the second auction, \( GSP_{2D} \), the allocation and pricing rules are defined to be exactly those of \( GSP \) when multiple ads are displayed. The idea behind defining the \( GSP_{2D} \) auction in this manner is that when multiple ads are displayed, advertisers see no difference at all between the new auction and the existing \( GSP \) system.

First, we develop and analyze a simple game-theoretic model which provides several insights into exclusive-display auctions. We show that allowing the advertisers to bid for exclusivity can generate higher revenue for the search engine. However, it can also create new set of equilibria with significantly lower revenue. Our results also make it clear that search engines should not adopt auctions with exclusive placement only; rather, they should adopt hybrid auctions that allow advertisers to bid for multiple as well as exclusive placement. While exclusive display can be the outcome and can provide higher revenue for the search engine when an advertiser highly values being the only one displayed, interestingly, we find situations in which search engine revenue increases even when the outcome is multiple display (and not exclusive display), simply because advertisers could bid for exclusive display also. In fact, the point at which the search engine is indifferent between multiple display and exclusive display, it can extract all of the bidders’ surplus as its revenue. Under other situations, both search engine revenue and bidders’ surplus can simultaneously increase.

We also derive results regarding which auction, \( NP_{2D} \) or \( GSP_{2D} \), gives higher search engine revenue and allocative efficiency under different conditions. In general, we find that \( NP_{2D} \) is the better auction in terms of revenue as well as allocative efficiency, which lends support to the idea that the simple “next price” heuristic of \( GSP \) is a good heuristic to use while designing extended position auctions.

Next, we run a comprehensive simulation study in which we analyze more realistic, and more complicated, situations. The simulations confirm the results from our simpler analytical model and also provide new insights. For instance, we find that a larger number of bidders significantly increases the revenue advantage to the search engine of using two dimensional auctions. We also find that as heterogeneity among advertisers in their valuations for exclusive display increases, both search engine revenue and bidders’ surplus can be significantly higher in two-dimensional auctions as compared to the one-dimensional \( GSP \) auction.

There is keen interest from the industry in exclusive display in sponsored search advertising, but there is lack of good understanding of the same, as is clear from the Google episode (Metz 2011). In this context, our study is very timely and relevant. Our insights into exclusive-display auctions can help to inform search engines as well as advertisers about the advantages and drawbacks of this new display format in the evolving sponsored search industry. For instance, Google’s dilemma seems to have been that allowing exclusive display might reduce revenue because of the loss of revenue from the many advertisers who will not be displayed. We find that, in fact, allowing advertisers to bid for both multiple and exclusive display can significantly increase search engine revenue. First, even if there is one advertiser who values exclusive display highly for a keyword (e.g., for its own brand name), then this advertiser will submit a large bid for exclusive display and exclusive display will be chosen as the auction outcome. Moreover, even if the outcome is multiple display, revenue
increases (compared to GSP) as long as some advertiser’s valuation for exclusive display is large, but not large enough to switch the outcome to exclusive display. In fact, we conjecture that multiple display with higher revenue than GSP will be the likely outcome for most keywords. This is because we can expect that exclusive display will be valued more than multiple display by almost all advertisers for almost all keywords, but may not be valued so much more that one advertiser dominates all others combined. In other words, we foresee that if a search engine adopts two-dimensional auctions with an exclusive display option for advertisers, the outcomes will still be multiple display for most keywords while the search engine will make greater revenue on these keywords.

The rest of this chapter is structured as follows. In the next section, we briefly review the literature related to our work. In Section 4.3 we develop the general framework for our analysis. In Section 4.4 we define and analyze a simple model within our general framework and obtain insights into the dynamics of exclusive-display auctions. In Section 4.5 we conduct simulations which numerically confirm the results from our analytical exercise in a more complicated setting and also provide additional insights. In Section 4.6 we conclude by summarizing our results and laying out directions for future work.

4.2 Related Literature

Theoretical studies in the Economics and Marketing communities have significantly enhanced our understanding of position auctions used in sponsored search advertising. Edelman and Ostrovsky (2007) studied first-price auctions and established that bidding will be cyclical and unstable in these auctions. Edelman et al. (2007) and Varian (2007) showed that bidding is stable in the Generalized Second Price auction (GSP), but bids do not truthfully reveal valuations of advertisers for positions. Various other papers that consider different aspects of second-price position auctions include Athey and Ellison (2011), Edelman and Schwarz (2010), Jerath et al. (2011), Katona and Sarvary (2010), Liu et al. (2010) and Desai et al. (2010). Many of the above papers consider both pay-per-impression and pay-per-click payment schemes. Zhu and Wilbur (2011) consider hybrid auctions in which advertisers can choose to bid on a per-impression or a per-click basis, while Agarwal et al. (2009) analyze bidding in a pay-per-action second-price auction. All of the above papers, however, study auctions that only consider displaying multiple advertisers in response to a keyword search.

To our knowledge, only two other papers (in the Computer Science community) analyze position auctions in which advertisers can express their preferences beyond simply turning in bids for a multiple-display outcome. Muthukrishnan (2009) considers a second-price auction and allows each advertiser to submit a per-click bid (its maximum willingness to pay) and specify the maximum number of other advertisers it wants to be displayed with. Note that this is a very different auction mechanism from the ones we consider in this essay. Furthermore, the focus of Muthukrishnan (2009) is on developing a fast algorithm to determine the outcome of this auction (which includes deciding how many ads to display, and which advertisers to include and how to rank them), while the revenue and efficiency properties of the auction itself are not analyzed. The paper closest to our work is Ghosh and Sayedi (2010), who analyze the same auctions as we do. However, they derive a very different set of results as their focus is on comparing the properties of the multiple equilibria that the
NP$^{2D}$ and GSP$^{2D}$ auctions can attain. In contrast, in this essay, our aim is to understand at an intuitive level how exclusive-display auctions work, and which auction is more beneficial to the search engine and to the advertisers under different conditions. We believe that our results and insights, while being of academic interest, also speak closely to the needs of a managerial audience.

Finally, there is previous work in Economics and Marketing that our work is related to. There is a small literature on multi-dimensional auctions in which bidders, differentiated on multiple characteristics, submit multi-dimensional bids and a winner is determined (Branco 1997, Che 1993, Mori 2006, Thiel 1988). For example, in an auction for a contract to build an aircraft, bidders quote a price and also specify the components of the aircraft (Branco 1997). Additionally, note that exclusivity contracts are often negotiated between media providers and advertisers for traditional media advertising. For example, Anheuser-Busch and Volkswagen held the rights for advertising exclusively in the beer and automotive categories, respectively, during Super Bowl 2011 and 2001. Dukes and Gal-Or (2003) study this market. However, our work is very different from these literature streams. First, the institutional details of our setting introduce several differences (e.g., ranked outcomes, per-click bidding by advertisers, bid-weighting by the auctioneer). Second, in our specific case the auction mechanism allows multiple as well as exclusive winners and the auctioneer decides after the bidders have submitted their bids whether there will be multiple winners with a rank ordering or only one winner.

4.3 Framework for Analysis

In this section, we describe the general framework that we use in the essay. When a user of the search engine submits a query, she is shown two lists of results—the organic list and the sponsored list. The sponsored list is a ladder of, usually text-only, ads towards the right of the results page. Sometimes, one to four ads are also placed above the organic search results. A position that contains an ad is called a slot, and the search engine basically assigns the ads to the slots. The slots that are placed above the organic search results are more likely to get clicks than those placed on the right and are considered more valuable; similarly, the slots placed at upper positions are more valuable than those placed in lower positions. Therefore, we get a total ordering, and we can model the ad presentation as an array of slots where the first positions in the array are more valuable and more likely to get clicks than the later positions.

We assume that there are $n$ advertisers who want to display their ads. In our context, ads can be displayed in one of two formats. In the first format, multiple ads are be displayed; specifically, $k$ ads can be displayed, where $k < n$. Slot $i$ is associated with a number $0 < \theta_i \leq 1$ called the click-through rate (CTR) of the slot. The number $\theta_i$ indicates the probability of being clicked if multiple ads are displayed and an ad is placed at slot $i$. According to the above discussion, $\theta_i$s are sorted in descending order along the array of slots. Previous research has found that click-through rates decreases exponentially with

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decending position (Feng et al. 2007). In accordance with this, we assume that \( \theta_i = p^{i-1}\theta_1 \) for \( i \in \{2, 3, ..., k\} \), where \( 0 < p < 1 \). We define \( \theta_i = 0 \) for \( i > k \).

In the second format, only one ad is displayed exclusively. In this case, we assume that the click-through rate of the only slot shown is \( \hat{\theta} \). We assume that \( \hat{\theta} \geq \theta_1 \), i.e., the only slot shown in the exclusive-display outcome gets at least as many clicks as the first slot in the multiple-display outcome. Since normalizing does not affect our results, we assume \( \hat{\theta} = 1 \) for simplicity.

We assume that there are \( n \) advertisers. Each advertiser \( i \) has a vector of valuations \((v_i^N, v_i^E)\) where \( v_i^N \) is the valuation of a click when displayed with multiple other advertisers and \( v_i^E \) is the valuation of a click when displayed alone in response to a keyword search (the superscripts \( N \) and \( E \) stand for non-exclusive and exclusive, respectively). For the reasons discussed in the introduction, it is reasonable to assume that \( v_i^E \geq v_i^N \). The search engine runs an auction in which it invites bids from advertisers. We now describe the different auction mechanisms.

### 4.3.1 One-Dimensional Auction: GSP

Major search engines, such as Google, Yahoo! and Bing, use a Generalized Second Price (GSP) auction for allocation and pricing. Consider a keyword for which each advertiser \( i \) submits a bid \( b_i \). The search engine sorts the advertisers in descending order of their bids and allocates the first (highest, and most valuable) slot to the highest bidder, the second slot to the second-highest bidder, and so on, until either all slots are allocated, or the bid is lower than the reserve price \( r \). (A reserve price of \( r \) means that the search engine would rather leave the slot empty than to sell it at a price less than \( r \).) Suppose that the bids submitted by the advertisers are \( b_1, \ldots, b_n \), and without loss of generality assume that \( b_1 \geq b_2 \geq \ldots \geq b_n \).

Therefore, as we described, the bidder \( i \) with bid \( b_i \) gets the \( i \)-th slot, as long as \( b_i \geq r \) and \( i \leq k \), where \( r \) is the reserve price and \( k \) is the number of slots available\(^2\).

The bidder who is assigned to the \( i \)-th slot has to pay \( \max(b_{i+1}, r) \) every time a user clicks on his ad; in other words, the payment rule is pay per click, and every bidder has to pay the minimum amount necessary to keep his position. For example, if bidder \( i \) whose bid is \( b_i \) changes his bid all the way down to \( \max(b_{i+1}, r) \), but not to any anything below that, he would still get the same slot; therefore, the price of bidder \( i \) is set to \( \max(b_{i+1}, r) \).

This characteristic makes the GSP a “next-price” auction. Note that since multiple ads are always displayed, the valuations of advertisers for exclusive display have no relevance in the GSP auction.

\(^2\)Search engines often transform a bid \( b_i \) to an effective bid \( \hat{b}_i = \gamma_i \times b_i \), where \( \gamma_i \) is a quality score which depends upon the past performance of the ad (how likely it is to generate clicks), the relevance of the ad to the keyword, the reputation of the advertiser, etc. The search engine only works with the effective bids rather than the original bids. The practice of transforming original bids into effective bids does not play an important role in the analysis presented in our work and does not change the key insights. Hence, for ease of understanding, we present our results without considering this transformation.
4.3.2 Two-Dimensional Auctions: \(NP_{2D}\) and \(GSP_{2D}\)

The goal of this chapter is to understand auctions which allow advertisers to bid for being shown exclusively on the results page. While bidding for exclusivity is not yet implemented at any major search engine, many of them are internally experimenting with such auctions. We analyze the two auction mechanisms, \(NP_{2D}\) and \(GSP_{2D}\), recently proposed and patented by Yahoo!. Since the current mechanism used by major search engines, including Yahoo!, is \(GSP\), both proposed mechanisms are also extensions of \(GSP\).

\(NP_{2D}\) and \(GSP_{2D}\) are “two-dimensional” auctions, i.e., each advertiser simultaneously submits two bids \(b^N\) and \(b^E\) where \(b^N\) indicates how much they are willing to pay per click if their ad is shown among other ads, and \(b^E\) indicates how much they are willing to pay per click if their ad is shown exclusively (as before, the superscripts \(N\) and \(E\) stand for non-exclusive and exclusive, respectively). The outcome of the auction can be either \(S\) or \(M\), where \(S\) means that only one ad is displayed, and \(M\) means that multiple ads are shown to the user (\(S\) stands for Single and \(M\) stands for Multiple). We call the non-exclusive bid of an advertiser, \(b^N\), his \(M\)-bid, and the exclusive bid, \(b^E\), his \(S\)-bid.

If the outcome is \(M\), we assume that \(k\) ads are shown and their click-through rates of the \(k\) ordered slots are \(\theta_1, \ldots, \theta_k\), where \(\theta_i = p^{i-1}\theta_1\) for \(i \in \{2, 3, \ldots, k\}\) and \(0 < p < 1\). If the outcome is \(S\), the click-through rate of the only slot shown is \(\theta = 1 \geq \theta_1\).

In the following subsections, we describe the mechanisms of \(NP_{2D}\) and \(GSP_{2D}\). We assume that the \(n\) bidders submit bids \((b^N_1, b^E_1), \ldots, (b^N_n, b^E_n)\), and without loss of generality we assume that \(b^N_1 \geq b^N_2 \geq \ldots \geq b^N_n\). To make the notation easier in the following sections, we assume that \(\max\) and \(\max_2\) are the indices of the bidders with the highest and the second highest \(S\)-bids, i.e., \(b^E_{\max} \geq b^E_i\) for any \(i\), and \(b^E_{\max_2} \geq b^E_i\) for any \(i \neq \max\). Also, let \(\max_{-i}\) be the index of the bidder who has the highest \(S\)-bid other than \(i\); therefore, for \(i \neq \max\) we have \(\max_{-i} = \max\) and for \(i = \max\) we have \(\max_{-i} = \max_2\). We assume that the reserve price set by the auctioneer is \(r\) for both non-exclusive and exclusive bids.

**The \(NP_{2D}\) Auction:** The first extension of \(GSP\) for the two-dimensional setting, \(NP_{2D}\), is based on the simple “next-price” rule of \(GSP\)—every winner has to pay the minimum amount necessary to keep his position. In \(GSP\), the next price is the bid of the next-highest bidder. In the two-dimensional \(NP_{2D}\), however, maintaining one’s position consists of two things for a winner in outcome \(M\): first, the outcome must remain \(M\) and not switch to \(S\); second, the bid must enable the bidder to maintain his position amongst the \(k\) slots. In a next price auction for our setting, therefore, the payment of a winner in slot \(i\) of outcome \(M\) is the larger of two terms—the first being the minimum value at which the outcome still remains \(M\), and the second being the bid of the next bidder, \(b^N_{i+1}\), as in \(GSP\). The \(NP_{2D}\) auction with reserve prices is formally defined below.

**The \(NP_{2D}\) auction** Let \(\Gamma = \sum_{i=1}^{k} \theta_i \max(b^N_i, r)\). The mechanism \(NP_{2D}\) compares \(b^E_{\max}\) to \(\Gamma\) to decide whether the outcome should be \(S\) or \(M\).

- If \(b^E_{\max} \geq \Gamma\), the outcome is \(S\) with payment
  \[
  \max(b^E_{\max}, \sum_{i=1}^{\max_2} \theta_i \max(b^N_i, r) + \sum_{i=\max}^{k} \theta_i \max(b^N_{i+1}, r)).
  \]


If $b_{\text{max}}^E < \Gamma$, the outcome is $M$ and the bidder winning slot $i \neq \text{max}$, if his bid is at least $r$, pays

$$\theta_i p_i = \max(\theta_i \max(b_{i+1}^N, r), b_{\text{max}}^E - \Gamma + \theta_i \max(b_i^N, r)),$$

while the bidder max, if his bid is at least $r$, winning slot $j (= \text{max})$ pays

$$\theta_j p_j = \max(\theta_j \max(b_{j+1}^N, r), b_{\text{max}}^E - \Gamma + \theta_j \max(b_j^N, r)).$$

The **GSP$_{2D}$ Auction**: The second extension of GSP for the two-dimensional setting, GSP$_{2D}$, is defined to ensure the practical benefit that when multiple ads are displayed, advertisers see no difference at all between the new auction and the existing GSP system. In other words, GSP$_{2D}$ is designed to restrict the allocation and pricing to be exactly those of GSP whenever multiple ads are shown (the outcome is $M$). It remains to decide whether to show outcome $M$ or outcome $S$, as well as the pricing and allocation for outcome $S$. A formal definition of these rules is as follows.

The **GSP$_{2D}$ auction** The mechanism GSP$_{2D}$ compares $b_{\text{max}}^E$ to $\sum_{i=2}^{k+1} \theta_{i-1} \max(b_i^N, r)$ to decide whether the outcome should be $S$ or $M$.

- If $b_{\text{max}}^E \geq \sum_{i=2}^{k+1} \theta_{i-1} \max(b_i^N, r)$, the outcome is $S$ with winning bidder max, whose payment is $\max(\sum_{i=2}^{k+1} \theta_{i-1} \max(b_i^N, r), b_{\text{max}}^E)$ per click.
- If $b_{\text{max}}^E \leq \sum_{i=2}^{k+1} \theta_{i-1} \max(b_i^N, r)$, assign the page to the bidders with $k$ highest M-bids as long as their M-bids are at least $r$, and charge them according to GSP pricing, i.e. bidder $i$ (if assigned any slot) has to pay $\max(b_{i+1}^N, r)$ per click.

Note that the highest M-bid is ignored in the above. Perhaps, the most natural choice would be to compare $b_{\text{max}}^E$ and $\sum_{i=1}^{k} \theta_i \max(b_i^N, r)$ for choosing between $M$ and $S$. However, one can easily see that with such rule, outcome $S$ almost never happens in equilibrium. If advertiser 1 is different from advertiser max, advertiser 1 can submit a very large M-bid to make sure that $\sum_{i=1}^{k} \theta_i \max(b_i^N, r) > b_{\text{max}}^E$, i.e., the outcome is $M$. Note that the payment of advertiser 1 is set by GSP in outcome $M$, i.e., he does not have to pay anything more than $b_2^N$ (the same as before) for submitting such a large bid. This bad equilibrium behavior is fixed by ignoring the highest M-bid.

### 4.4 Analytical Study

We build insights into the different auction mechanisms by considering a simplified analytical model which is a special case of the general framework described in the previous section. The insights derived from this simplified analytical exercise are valid in the fully general framework; we choose the simpler model because it allows us to bring to light the key insights through a tractable, closed-form analysis. In Section 4.5, we report the results of a numerical simulation study using the full model which confirms the results derived in this section.
4.4.1 Simplified Model with Ex-post Nash Equilibria

We assume that there are three bidders (i.e., \( n = 3 \)) labeled A, B and C with valuation vectors \((a, a), (b, b)\) and \((0, c)\), respectively, where \(a, b, c > 0\). In other words, the valuation per click of bidders A and B remains the same whether their ads are listed with other ads or listed exclusively. Bidder C, however, only values being listed exclusively. Without loss of generality, we assume that \(a \geq b\). We also assume that if the outcome is \(M\), there are two slots available (i.e., \(k = 2\)). Recall that we have already assumed that the CTR of the only slot in outcome \(S\) is 1. To be conservative in our evaluations of the two-dimensional auctions, we assume that the CTR of the first position in \(M\) is also 1. This implies that the CTR of the second position in \(M\) is \(p\), so that the total expected clicks in \(M\) are \(1 + p\), which is more than the total expected clicks in \(S\). We are only interested in the case that \(b \geq r\) which means that the bidders are both competitive enough and can beat the reserve price; otherwise, the bidder will be completely ignored by the mechanism.

Bidder \(i\) submits a two-dimensional bid \((b^N_i, b^E_i)\) where \(b^N_i\) is how much he bids for being shown among the others, and \(b^E_i\) is how much he bids to appear exclusively. We calculate the equilibrium revenue and efficiency of \(GSP\), \(GSP_{2D}\) and \(NP_{2D}\). However, there are multiple Nash equilibria for each of these mechanisms. This makes comparisons among the mechanisms very hard since, in most of the cases, all mechanisms have good and bad equilibria in terms of revenue and efficiency. Hence, we impose the following reasonable equilibrium refinements to reduce the number of equilibria that we obtain.

1. Losers bid at least their true value: This is particularly relevant when the outcome is \(S\); while there might be Nash equilibria where the losing bidders must bid \(b^N_i < v^N_i\) to ensure that the winner has no incentive to deviate to outcome \(M\), it is unreasonable to expect that the losing bidders will not bid higher in an effort to change the outcome to \(M\), which would give them positive utility. Thus, Nash equilibria where the outcome is \(S\) but losers bid less than their true values simply to maintain equilibrium are unlikely to exist in practice.

2. Winners bid the lowest in their best-response set: If two different bids result in the same outcome for a winner, we assume that he chooses the lower one. In other words, as long as a bidder is getting his desired outcome, he prefers to bid the lowest value that guarantees him the same outcome.

Moreover, we assume throughout the chapter that the bidders do not play weakly dominated strategies. The above refinements allow us to make easier analytical comparisons among the mechanisms in terms of revenue and efficiency. We also assume that the ties are broken in favor of the stronger bidder (bidder with higher valuation), and between \(S\) and \(M\) we assume that the ties are broken in favor of outcome \(S\).

4.4.2 Analysis with Ex-post Nash Equilibria

In this section, we derive the equilibrium revenue and social welfare for the different auctions in three lemmas. The proofs accompanying each lemma provide insights into the working of each auction.
Lemma 4.4.1. The revenue of GSP is \( r + rp \), and its social welfare is \( a + bp \) or \( b + ap \).

Proof. Note that this is a one-dimensional mechanism and hence each bidder submits only one bid. It is weakly dominated for bidder C to bid more than 0, so we assume that his bid is 0. Let the highest and the second highest bids be \( x \) and \( y \) respectively. Clearly, both \( x \) and \( y \) should be at least \( r \); moreover, the bidder who is getting the second slot would still get the second slot if he bids \( r \) (and not below \( r \)), therefore, \( y = r \). By the definition of GSP it is easy to see that the revenue of the mechanism in this case is \( r + rp \). Note that \( x \) should be high enough so that the bidder who is getting the second slot with utility \( p(v - r) \) where \( v \) is either \( a \) or \( b \) does not want to deviate and get the higher slot; the minimum value of \( x \) satisfying this condition is \( x = pr + v - pv \). □

Note that there are multiple equilibria for GSP, but all of them lead to the same revenue \( r + rp \). In case of welfare, we will show that the welfare of two-dimensional auctions is generally better than the one-dimensional auction. To favor the one-dimensional auction in this comparison, we pick the equilibrium of GSP which has the higher welfare, \( a + bp \).

Lemma 4.4.2. The revenue of NP2D is \( \min(\max(c, r + rp), a + bp) \) and its social welfare is \( \max(c, a + bp) \).

Proof. Let the equilibrium vector of bids be \((x, x'), (y, y')\) and \((z, z')\); without loss of generality assume that \((z, z')\) is bidder C’s bids and let \( x \geq y \). First consider the case where \( x + yp \geq z' \). In this case, bidder C is a loser and hence \( z' \geq c \). Furthermore, it is enough for \( x + yp \) to be \( c \) to make the outcome M; so since bidders A and B minimize their bids, we have \( x + yp \leq \max(c, r + rp) \). Also, note that since the bidders do not have negative utility, we must have \( a + bp \geq c \). Furthermore, if \( a + bp \geq c \), bidder C cannot be the winner of S in any case, because bidders A and B can always change the outcome to M by bidding truthfully. Finally, note that if C is not the winner of S, no other winner could be the winner of S, either: this is because the winner should pay at least \( b + rp \), however, if he deviates and changes the outcome to M, he would pay at most \( p \) for the same CTR. Therefore, we know that if \( c < a + bp \), the outcome is M, and the revenue is \( \max(c, r + rp) \).

Now consider the case where \( c > a + bp \). In this case, bidder C will be the winner because otherwise, he would S-bid at least \( c \) and this makes the utility of at least one of the two other bidders negative. Given that C is the winner of S, the other two bidder bid at least their true values, hence the revenue of the mechanism is \( a + bp \). Since \( r + rp \leq a + bp \), based on the two cases that we discussed, the revenue is always \( \min(\max(c, r + rp), a + bp) \). □

Lemma 4.4.3. If \( c \geq a + rp \), then the revenue of GSP2D is \( \max(b + rp, a) \) and its social welfare is \( c \). If \( c < a + rp \), the revenue of GSP2D is either \( \max(c, b, r + rp) \) or \( \max(c, r + rp) \) and its social welfare is \( a + bp \) or \( a \), respectively.

Proof. Let the equilibrium vector of bids be \((x, x'), (y, y')\) and \((z, z')\); without loss of generality assume that \((z, z')\) is bidder C’s bids and let \( x \geq y \). First consider the case where \( z' \geq y \) and \( z' \geq \max(x', y') \): the outcome is S. Therefore, bidders A and B are both losers, and hence \( y \geq b \) and \( x \geq a \). Moreover, B should not be able to make a profitable deviation that changes the outcome to M, i.e., even if \( y \) is increased up to \( a \), the outcome should still...
remain S. Therefore, it must be the case that \( z' \geq a + rp \) which requires \( c \geq a + rp \). By definition of GSP\(_{2D}\), the revenue is \( \max(b + rp, a) \).

Now, consider the case where \( z' \geq y \) but \( z' < \max(x', y') \): the outcome is still S; however, the winner is not bidder C anymore. First, note that bidder B cannot be the winner because otherwise, bidder A would be a loser and hence bids at least \((a, a)\) which makes the price for bidder B at least \( a \) meaning that her utility is negative. So, bidder A must be the winner. In this case, his price is \( \max(\max(c, b), r + rp) \), therefore, it must be the case that \( c \leq a < a + rp \) since otherwise the utility of bidder A would be negative. The revenue is \( \max(c, b, r + rp) \) as claimed.

Finally, consider the case where \( z' < y \). Clearly, bidder C cannot be the winner in this case. Therefore, his S-bid is at least \( c \). The outcome is either S or M. First suppose that the outcome is S: the winner should be bidder A because otherwise, A would be a loser which makes his S-bid at least \( a \) which makes the utility of bidder B negative. Since the outcome is S, bidder B should be M-bidding and S-bidding at least \( b \). Therefore, the price for bidder A, and consequently the revenue of the mechanism, is \( \max(c, b, r + rp) \). Next, suppose that the outcome is M: the winners should be bidders A and B. Therefore, bidder C bids at least \( c \) which means \( y > c \). We also know that \( y \) should be at least \( r \) for the bidder to meet the reserve price and win the slot. As long as \( y \geq \max(r, c - rp) \), the outcome is M, and since we already know that \( y \geq c > c - rp \), the minimum value that the bidder can bid and still win the same outcome (the second slot here) is \( r \). Therefore, the payment of the bidder with the higher bid \( x \) is \( = r \), and the payment of the bidder with lower bid \( y \) is \( rp \). Consequently, the revenue is \( r + rp = \max(c, r + rp) \), as desired.

Note that there are multiple equilibria in some cases for GSP\(_{2D}\). In particular, if \( c \) is small, there are two possible equilibria: (i) the outcome is M, and A wins the first slot and B wins the second slot, and (ii) A wins S. The revenue of the first equilibrium is \( \max(b, c, r + rp) \), while the revenue of the second equilibrium is \( \max(c, r + rp) \).

### 4.4.3 Bayesian Nash Equilibria

The equilibrium concept defined in Section 4.4.1 and discussed so far is ex-post Nash equilibrium. We showed that under the refinement defined in Section 4.4.1, search engine always benefits from allowing exclusive bidding. In this section, we use Bayesian Nash equilibrium concept to improve our understanding of the dynamics of the game. Interestingly, we show that allowing exclusivity can lead to a new set of equilibria with significantly lower revenue for the search engine.

Similar to Section 4.4.1, we assume that there are three bidders (i.e., \( n = 3 \)) labeled A, B and C. However, we generalize the valuation vectors to \((a, a'), (b, b')\) and \((c, c')\), respectively, where \( a, b, c > 0 \) and \( a' \geq a, b' \geq b \) and \( c' \geq c \). In other words, all three bidders weakly prefer being shown exclusively. We also assume that if the outcome is M, there are two slots available (i.e., \( k = 2 \)). Recall that we have already assumed that the CTR of the only slot in outcome S is 1. We assume that the CTR of the \( i \)-th position in M is \( p_i \).

As in Section 4.4.1, bidder \( i \) submits a two-dimensional bid \((b_i^N, b_i^E)\) where \( b_i^N \) is how much he bids for being shown among the others, and \( b_i^E \) is how much he bids to appear exclusively.
We assume values $a, a', b, b', c$ and $c'$ are drawn from an arbitrary joint distribution. Finally, we assume that $p_2 \leq p_1 < 1$.

**Theorem 4.4.4.** Bidder $i$ (for $i \in \{A, B, C\}$) submitting $(0, v_i)$ where $v_i$ is his true valuation for exclusivity is an equilibrium for $NP_{2D}$ and $GSP_{2D}$.

*Proof.* Note that in the mentioned equilibrium, bidders are essentially participating in a second price auction. Hence, bidding above or below true valuation for exclusivity is dominated, as long as the bidder is not bidding for non-exclusive outcome. It remains to show that they cannot benefit from bidding non-zero value for non-exclusive outcome.

Note that since exclusive valuation is always greater than or equal to non-exclusive valuation, with only one non-exclusive bid the outcome cannot be non-exclusive. Therefore, even if one bidder deviates and submits non-zero bid for non-exclusive outcome, the outcome will still be exclusive. However, if this bidder wins the exclusive outcome, he may have to pay more than when bidding 0 for non-exclusivity. Therefore, bidding non-zero for non-exclusive outcome can only decrease a bidder’s expected payoff. □

Theorem 4.4.4 shows that it may happen that in equilibrium, all bidders bid only for exclusive outcome. This equilibrium could clearly have lower revenue for the search engine than the ones in $GSP$. In particular, if $p_1 + p_2 > 1.5$ and exclusive valuations are the same as non-exclusive valuations, $GSP$ will have higher revenue than the equilibrium in which every bidder bids only for exclusivity.

Finally, it is worth mentioning that the equilibrium in which all bidders bid only for exclusivity is also an ex-post Nash equilibrium; however, if treated as an ex-post equilibrium, it does not satisfy the refinement conditions in Section 4.4.1.

### 4.4.4 Results and Insights

A representative comparison among the revenues of the auctions as calculated in Section 4.4.2 is shown in Figure 4.1(a). The figure shows that the revenue of either $NP_{2D}$ or $GSP_{2D}$ always dominates the revenue of $GSP$. Therefore, in ex-post Nash equilibrium, it is always better for the search engine to use a two-dimensional auction allowing exclusive display instead of the currently prevailing one-dimensional GSP auction. Figure 4.1(a) shows that, generally speaking, $NP_{2D}$ has higher revenue; however, the revenue of $GSP_{2D}$ might be better than the revenue of $NP_{2D}$ for small values of $c$. (Note that $c < a + rp$ is the region in which $GSP_{2D}$ has multiple equilibria. The dashed green line corresponds to the higher-revenue equilibrium; the revenue of the other equilibrium of $GSP_{2D}$ is not necessarily more than the revenue of $NP_{2D}$.) We state this below as a proposition.

**Proposition 4.4.5.** In ex-post Nash equilibrium, either $GSP_{2D}$ or $NP_{2D}$ always provides higher revenue than GSP. If $b > r + rp$ then if $c$ is below a threshold value (specifically, $c < b$) then $GSP_{2D}$ can provide highest revenue for the search engine, and if $c$ is above this threshold value then $NP_{2D}$ provides highest revenue for the search engine. If $b < r + rp$ then $NP_{2D}$ always provides highest revenue for the search engine.

There are two primary reasons why exclusive-display auctions can provide higher revenue for the search engine. First, $NP_{2D}$ and $GSP_{2D}$ allow advertisers to express their valuations
Figure 4.1: Revenues, social welfare and fraction of social welfare extracted as revenue as functions of $c$ with parameters $p = 0.6$, $a = 3$, $b = 2$ and $r = 1$. The red line represents $NP_{2D}$, the solid green line represents $GSP_{2D}$ (the dashed green line represents the higher-revenue equilibrium of $GSP_{2D}$, defined only for $c < a + rp$), and the blue line represents $GSP$. The shaded region in panel (a) shows the values of $c$ below which the outcome of $NP_{2D}$ is M.
for exclusive display while GSP ignores these valuations. Clearly, if the valuation for exclusivity is large enough even for one advertiser, allowing this advertiser to express this valuation through his bid will switch the outcome to S from M and increase search engine revenue. This is the case when the value of $c$ is large.

However, there is a second, and more interesting, phenomenon at play which leads to higher revenue in auctions allowing exclusive display. Note that even when the outcome of $NP_{2D}$ is M (which happens if $c < a + bp$, which is the shaded region in Figure 4.1(a)), we can still see higher revenue in $NP_{2D}$ than in GSP. In other words, even if exclusive display is not the equilibrium outcome, auctions allowing bidders to bid for exclusive display can provide higher revenue to the search engine than the GSP auction. In fact, we can see that the M outcome in $GSP_{2D}$ can also generate higher revenue for the search engine than GSP, even though in this case, by the definition of $GSP_{2D}$, the bidders will be ranked in $GSP_{2D}$ exactly as they would be ranked in GSP.

To understand the intuition behind this, consider the case in which there is a bidder who values the exclusive-display outcome more than the multiple-display outcome, but this valuation for exclusivity is not too high (which is true for medium values of $c$). In this case, the other bidders who value the M outcome close to the S outcome have to bid higher than they do in GSP to actually keep the outcome as M (i.e., to prevent the outcome from becoming S, in which case they will not be displayed and be worse off), which leads to higher revenue. Said in another way, two-dimensional auctions give more degrees of freedom to the bidders which also increases the competition among them. This increased competition, in equilibrium, leads to higher revenue for the search engine. We state this below as a proposition.

**Proposition 4.4.6.** Even if the outcome of a two-dimensional auction ($NP_{2D}$ or $GSP_{2D}$) allowing for exclusive display of an ad is such that multiple ads are actually displayed, it can generate higher revenue for the search engine than the GSP auction due to greater competition among the bidders.

On the other hand, allowing exclusive bidding can also decrease search engine’s revenue. Two-dimensional bidding creates new set of equilibria some of which have lower revenue than GSP. As discussed in Section 4.4.3 all bidders bidding only for exclusivity is an example of low-revenue equilibria which appear when exclusive bidding is allowed.

Note that an interesting property of the revenues of $NP_{2D}$ and $GSP_{2D}$ auctions is that $NP_{2D}$ is revenue monotone in valuation for clicks while $GSP_{2D}$ is not. In other words, if the valuation of a bidder for clicks increases, the revenue of $NP_{2D}$ will not decrease, while this is not true for $GSP_{2D}$. In fact, from Figure 4.1(a) it is clear that the revenue of $GSP_{2D}$ drops down at some point as $c$ increases.

Another interesting and important metric to study for these auctions is the social welfare, which is the sum of the search engine profits and the bidders’ surplus. To study this, we define a new two-dimensional auction called $VCG_{2D}$. $VCG_{2D}$ is an extension of the one-dimensional social-welfare-maximizing $VCG$ auction, and maximizes social welfare in the two-dimensional setting because the outcome is chosen as S or M and slots are subsequently allocated to the advertisers based on who values them more. We define and analyze $VCG_{2D}$ in the appendix, and find that its social welfare is given by $\max(c, a + bp)$. Interestingly, in ex-post equilibrium, this is exactly the social welfare of the $NP_{2D}$ auction. This implies
that, for our simplified model with ex-post equilibrium, \( NP_{2D} \) is a social-welfare-maximizing auction. The other auctions, \( GSP \) and \( GSP_{2D} \), also achieve the maximum social welfare for some regions of the parameter space, but there are other regions of the parameter space where they do not. This is shown with the help of Figure 4.1(b).

Although \( NP_{2D} \) and \( VCG_{2D} \) are both social-welfare maximizing in ex-post equilibrium, only \( VCG_{2D} \) remains social-welfare maximizing if we use Bayesian Nash equilibrium, discussed in Section 4.4.3 In Bayesian Nash equilibrium, truthful bidding (in both dimensions) is dominant strategy for \( VCG_{2D} \). However, equilibrium welfare of \( NP_{2D} \) and \( GSP_{2D} \) could be significantly lower than that of \( VCG_{2D} \). Furthermore, in equilibrium defined in Theorem 4.4.4, welfare of \( NP_{2D} \) and \( GSP_{2D} \) can be even lower than welfare of \( GSP \).

In ex-post equilibrium, for small and medium \( c \), compared to \( GSP \), both \( NP_{2D} \) and \( GSP_{2D} \) can increase search engine revenue without decreasing social welfare, which implies that two-dimensional auctions help the search engine to extract more from the bidders. Furthermore, for large \( c \), the social welfare increases, and both search engine revenue and bidders’ surplus increase simultaneously. These patterns are clear if we inspect Figures 4.1(a) and 4.1(b) simultaneously. We state the interesting results in the following proposition.

**Proposition 4.4.7.** Two-dimensional auctions can simultaneously increase both search engine revenue and bidders’ surplus.

Next, we investigate at how good the two-dimensional setting performs in extracting as much revenue as possible. For this, we use the fact the the revenue of a mechanism in equilibrium can never be more than the social welfare, otherwise, some bidders would incur negative utility which is against the definition of equilibrium and individual rationality. Therefore, the ratio of search engine revenue to the maximum possible social welfare (obtained from the \( VCG_{2D} \) auction) is a measure of how good an auction is at extracting as much revenue as possible.

Figure 4.1(c) plots the fraction of the maximum possible revenue extracted by the various auctions considered. It shows that two-dimensional auctions not only extract more revenue than \( GSP \) in absolute value, but also the fraction of the total possible revenue extracted is larger than the fraction extracted by \( GSP_{2D} \). By looking at the curve of the fraction of possible revenue extracted by \( NP_{2D} \), we see that the curve increases up to the value 1 (which is the point \( c = a + pb \)), which implies that at this point \( NP_{2D} \) extracts all of the social welfare as profit for the search engine. Subsequently, as \( c \) increases, this fraction starts to decrease. The interpretation is as follows. As \( c \) increases from a small value, the competition between bidder C who wants outcome S and bidders A and B who want outcome M becomes more intense, and the increased competition makes the revenue larger. The point at which the search engine can extract all of the social welfare as its revenue is the point at which its revenue from both M and S is the same and it is indifferent between choosing M or S. However, as \( c \) increases further so that it becomes large enough such that the optimal outcome for the search engine is S, the extracted revenue remains constant with increasing \( c \). This is because, due to the next-price characteristic of the auctions, the revenue

\[^3\text{Note that two-dimensional auctions extract a larger fraction of the total possible revenue even when, to calculate the fraction for GSP, we use the social welfare of the one-dimensional setting as the denominator. This comparison normalizes the effect that only two-dimensional auctions (but not GSP) benefit from increasing c. Still, we see that NP}_{2D} \text{ and GSP}_{2D} \text{ perform better.}\]
is determined by the bids of the other advertisers who do not value exclusivity highly, while the maximum possible revenue, which is now \( c \), keeps increasing. Therefore, the ratio of the extracted revenue to the maximum possible revenue becomes smaller. We state this result as a proposition.

**Proposition 4.4.8.** The search engine can use two-dimensional auctions to extract a larger fraction of the social welfare as its profit (as compared to the one-dimensional auction GSP). Furthermore, under the condition \( c = a + bp \), when the search engine is indifferent between the outcomes M and S, NP\(_{2D}\) extracts all of the social welfare as profit for the search engine.

From the results above, we see that if \( c > a + bp \), then as it becomes larger, the search engine only makes the profit \( a + bp \) and therefore also extracts a progressively smaller fraction of the social welfare. (In the next section, we will see that having multiple bidders with large per-click valuations for exclusive placement mitigates this problem to a large extent.) However, a solution to this problem is to move to a new mechanism which only allows outcome S. This mechanism invites only one-dimensional bids for outcome S and dominates NP\(_{2D}\) in terms of revenue for large values of \( c \) by setting a high-enough reserve price. Therefore, interestingly, as \( c \) becomes very large, it is better for the search engine to use a one-dimensional exclusive-only bidding mechanism.

Overall, we can summarize the insights from our analytical study as follow. First, two-dimensional auctions that allow advertisers to bid for exclusive display in the sponsored search section in response to a user query can increase or decrease search engine’s profits depending on equilibrium selection. Search engine’s profit may increase even if the equilibrium outcome is to actually display multiple ads. Similarly, two-dimensional auctions can increase or decrease social welfare. Furthermore, how social welfare is split between the search engine and the advertisers depends on the value attached to exclusive display by advertisers. In fact, the NP\(_{2D}\) auction can appropriate all of the welfare for the search engine and leave the bidders with zero surplus, which the search engine is unable to achieve using the one-dimensional GSP auction. Under other conditions, the surplus of the bidders can also increase. Finally, the good revenue and efficiency properties of NP\(_{2D}\), compared to GSP\(_{2D}\), suggest that the simple “next price” heuristic of GSP is a good heuristic for designing two-dimensional position auctions as well.

### 4.5 Simulation Study

In this section, we conduct simulations for exclusive-display auctions that reflect more realistic situations (as compared to our simplified analytical model). We use ex-post Nash equilibrium concept for the simulations. The simulations support the results from our analytical study and also add new insights.

#### 4.5.1 Design of the Study

We use the general model defined in Section 4.3 and assume that there are \( n = 10 \) advertisers and \( k = 5 \) slots in the non-exclusive outcome. We assume that the CTR of the first slot in
Figure 4.2: A uniform distribution to sample $v_j^N$ and a triangular distribution to sample $h_j$, where $v_j^E = h_j v_j^N$. Smaller values of $a$ imply larger mean and variance (heterogeneity) in the exclusive-placement valuations.

the non-exclusive outcome is $\theta_1 = p$, $0 < p \leq 1$. This implies that the CTR of slot $i$ in the non-exclusive outcome is $p^i$, $i \in \{1, 2, 3, 4, 5\}$, which in turn implies that the total expected clicks in the non-exclusive outcome are $\sum_{i=1}^{5} p^i$. We use three values of $p$ in our simulations, specifically, 0.55, 0.7 and 0.85, which implies that the total number of clicks in the non-exclusive outcome are 1.16, 1.94 and 3.15, respectively. Recalling that the CTR of the only slot in the exclusive outcome is normalized to $\hat{\theta} = 1$, we note that, for the chosen values of $p$, the total number of clicks in the non-exclusive outcome, compared to the exclusive outcome, are about the same, about double and about triple, respectively.

The valuation of each advertiser $j$ for the non-exclusive outcome, $v_j^N$, is drawn uniformly randomly between [0,1]. The valuation of each advertiser for the exclusive outcome is set to $v_j^E = h_j v_j^N$, where $h_j \geq 1$ is a “heterogeneity multiplier” and is set in the following manner. For some given parameter $a > 0$, a value $h_j$ is drawn independently randomly for each advertiser with probability distribution defined by $Pr(h_j > 1+x) = (2-ax)^2/4$, $0 \leq x \leq 2/a$. This implies that $h_j \in [1, 1+2/a]$ and the probability density function for $h_j$ is a right-angled triangle with height $a$ at 1 and height 0 at $1+2/a$. We show this in Figure 4.2. Given this formulation, a smaller value of $a$ implies that the value of $h_j$ can be drawn from a wider domain. In other words, a smaller value of $a$ implies that the heterogeneity among the advertisers’ valuations for the exclusive slot is greater and the exclusive slot is also valued more on average. We use 39 different values of $a$ between 3 and 0.01 (shown in Table 4.1), which implies that the domains of $h_j$ vary between [1, 1.67] (small heterogeneity) and [1, 201] (large heterogeneity), respectively. We set the reserve price of this auction to $r = 0$.

Given the three values of $p$, and the 39 values of $a$, we get 117 different scenarios. For each scenario, we simulate the outcomes of the $GSP$, $NP_{2D}$ and $GSP_{2D}$ auctions 100 times. Before proceeding further, we note that the main differences between the analytical study and the simulation study are that there are more bidders and each bidder has randomly-drawn valuations for the non-exclusive and exclusive outcomes from the same distributions.
4.5.2 Computing the Equilibrium

We adopt and extend the procedure in Cary et al. (2008) to find the equilibrium in a given auction. This procedure is as follows. We start with all bids, exclusive and non-exclusive, being zero. At each step, an advertiser is selected uniformly at random. This advertiser updates his strategy by playing best response to the other advertisers’ current bids. This best response is found by a brute force technique—the advertiser tries different bids that will determine the configuration and put him in the different slots available in the configuration, and chooses the bid that maximizes his payoff. If he does not win any slot when playing his best response strategy, we make him bid at least his true value. (The reason for this is that we want to avoid those equilibria in which some bidders are losing but still bidding below their true valuation to maintain the equilibrium. Note that, in this situation, bidding his true valuation belongs to the set of best response strategies of the losing advertiser.) We continue repeating the above step until we reach an equilibrium, in which case no advertiser wants to update his strategy any further and all losers are bidding at least their true value.

4.5.3 Results and Insights

We summarize the results for the simulations with $p = 0.7$ in Table 4.1 and Figure 4.3. In Table 4.1, the column titled “h” shows the domain of $h$ for the corresponding value of $a$ in the leftmost column. For $NP_{2D}$ and $GSP_{2D}$, the column titled “S%” shows what percent of time the auction outcome is the exclusive outcome, the column titled “Rev” shows the revenue from the auction, the column titled “Welf” shows the social welfare from the auction and the column titled “Frac” shows the fraction of revenue to social welfare, each quantity averaged over 100 runs. (The “Frac” column for the $GSP$ auction is calculated by dividing the revenue of $GSP$ by the social welfare of $NP_{2D}$.)

The results illustrate that two-dimensional auctions—especially $NP_{2D}$—give greater revenue to the search engine than $GSP$. Moreover, this greater revenue is obtained even if the outcome of the two-dimensional auctions is non-exclusive display, for the reasons explained in Section 4.4. The social welfare is higher than $GSP$, and the fraction of social welfare extracted as revenue is also higher. These results get stronger as the value of $a$ decreases (going down the rows in Table 4.1).

An interesting observation is that, for large $a$, the exclusive outcome is obtained more often for $GSP_{2D}$. The reason is that, the manner in which $GSP_{2D}$ is defined, it ignores the highest non-exclusive bid. For large $a$, this is actually a disadvantage because it unnecessarily prefers the exclusive outcome, even when the non-exclusive outcome is the better one for the search engine in this situation. $GSP$, on the other hand, ignores the exclusive bids completely. $NP_{2D}$ does not suffer from either problem, and always provides the highest revenue. As $a$ decreases, the valuations for exclusivity (and therefore the bids for exclusivity) are high enough that it does not hurt to ignore the highest non-exclusive bid; therefore, $GSP_{2D}$ has better revenue properties for small $a$.

Another interesting point to note here is that our analytical study showed that $GSP_{2D}$ has two equilibria when valuations for the exclusive outcome are small, and one of these

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4The results from simulations with $p = 0.55$ and $p = 0.85$ are qualitatively similar and we omit them from the essay. They can be obtained from the authors upon request.
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<th>$a$</th>
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<td></td>
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<td>1.1 1.5 10%</td>
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<td>100% 68.86 97.51 71%</td>
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Table 4.1: Outcomes of auctions for $p = 0.7$.
Figure 4.3: Revenues, social welfare and fraction of social welfare extracted as revenue as functions of $c$ with parameter $p = 0.7$. $NP_{2D}$ is red, $GSP_{2D}$ is green and $GSP$ is blue. The $x$-axis is not to scale.
equilibria has higher revenue than \( NP_{2D} \) (Proposition 4.4.5). Our simulation study, however, consistently shows higher revenue for \( NP_{2D} \) on average. Upon deeper investigation, we find that the higher-revenue equilibrium of \( GSP_{2D} \) indeed occurs in the simulation, but does not occur sufficiently often to lead to higher average revenue than \( NP_{2D} \). This indicates that, in general, \( NP_{2D} \) might be the better choice of auction in terms of revenue.

Note that in the simulation setup there is more competition among bidders (than in the simplified analytical setup in Section 4.4). This is because in the model in Section 4.4, two bidders (A and B) have the same per-click valuation for multiple and exclusive display and only bidder C values exclusive display more than multiple display. In the simulation, each bidder has the pair of per-click valuations drawn from the same distributions (which implies that all bidders value exclusive display more than multiple display, albeit to different extents) and there are more bidders.

This greater competition manifests itself in two main differences in the outcomes as shown in Figures 4.1 and 4.3. To facilitate these comparisons, note that, loosely speaking, a smaller value of \( a \) has the same effect on the valuation for the exclusive outcome as a larger value of \( c \) in Section 4.4. First, note from the revenue plots that as \( a \) decreases the revenues increase sharply from the two-dimensional auctions in Figure 4.3(a), which does not happen as \( c \) increases in Figure 4.1(a). This is because, in the simulation, there are many bidders with high valuations for the exclusive outcome, not just one. This implies that the winning bid does not just have to beat the non-exclusive bids, but also other comparable exclusive bids. Therefore, the exclusive bids are higher and so is the revenue. Second, note from the plots showing the fraction extracted that the peak in Figure 4.3(c) is less pronounced than the peak in Figure 4.1(c). The peak occurs at the point at which the search engine is indifferent between choosing outcome M or S. With many bidders in the simulated auctions, the fraction of social welfare extracted as revenue is quite large for all values of \( a \). Therefore, the peak is not very distinct. However, as \( a \) decreases, the fraction of social welfare extracted as revenue from \( GSP \) falls sharply because it completely ignores exclusive-placement valuations.

Overall, observations from the simulation study parallel those from the analytical study. In fact, allowing for many bidders makes an even stronger case for two-dimensional auctions because the revenue and the fraction of social welfare extracted as revenue are both consistently significantly higher than in \( GSP \).

### 4.6 Discussion

Most search engines run auctions to price the ranked list of ads presented to a user in response to a keyword search. In the last decade, the type of auction used has evolved from a Generalized First Price auction to a Generalized Second Price (\( GSP \)) auction with numerous small adjustments, and search engines continue to explore new auction mechanisms that can improve revenue. Recently, the three largest search engines, namely Google, Yahoo! and Bing, have been experimenting with the idea of allowing an advertiser to bid to display his ad exclusively rather than in a list of multiple ads. Advertisers can be expected to value exclusive display more because they can create strong branding effects by being displayed exclusively with a keyword, and avoid negative externalities that may impact post-click actions of consumers due to the presence of other ads in a ranked list.
In this essay, we study two auctions, \(NP_{2D}\) and \(GSP_{2D}\), recently patented by Yahoo! as the primary candidates for implementation as exclusive-display auctions. In these auctions, each advertiser submits a two-dimensional bid, one for a multiple-display format and another for an exclusive-display format. We find that these exclusive-display auctions can generate higher or lower revenue than the \(GSP\) auction, depending on equilibrium selection. It is easy to see why revenue increases if some advertiser values exclusive display very highly—he will submit a high exclusive-display bid and the search engine will choose the higher-revenue exclusive display outcome. Interestingly, however, search engine revenue can increase even if the final outcome is multiple display. This is because advertisers are competing not only for ranks within the multiple-display outcome, but are also competing for the outcome itself. Therefore, if an advertiser values exclusive display more than multiple display, other advertisers who want the multiple display outcome (because they themselves cannot bid highly for being displayed exclusively, and will get nothing if someone else gets displayed exclusively) will have to bid higher than in \(GSP\) to make the multiple-display outcome more profitable for the search engine. On the other hand, exclusive-display auctions create new set of equilibria some of which may have very low revenue for the search engine. An example of such equilibria is when all advertisers bid only for exclusive outcome—no advertiser can single-handedly change the outcome to \(M\), and search engine’s revenue may decrease because fewer clicks are sold. We also run numerical simulations which show to confirm the results of our model.

We conjecture that if search engines adopt two-dimensional exclusive display auctions then, for a large majority of keywords, the outcome will still be multiple display while the search engine will make higher profits than in \(GSP\). This is because, for most keywords, advertisers can be expected to value exclusive display more than multiple display which will lead to the competitive effect explained above, but one advertiser may not value exclusive display so much that his bid by itself can dominate the total potential revenue from multiple advertisers combined. For other keywords, for instance, brand names, one advertiser may indeed value exclusive display high enough than others that the outcome can very well be exclusive display. Such insights can help to resolve the dilemma that search engines face regarding whether or not to adopt exclusive-display auctions (Metz 2011).

Among the \(NP_{2D}\) and \(GSP_{2D}\) auctions, we find that \(NP_{2D}\), which is a next-price auction, has better revenue and allocative efficiency properties. This suggests that the simple next-price heuristic of the one-dimensional \(GSP\) is a good heuristic for designing two-dimensional position auctions as well.

Our work is one of the first to model exclusive display in sponsored search advertising, and there are numerous avenues for future research. First, we analyze auctions that have been proposed as candidates for implementation at Yahoo!. One of these auctions, \(NP_{2D}\), has good revenue properties and can even extract the full social welfare as search engine revenue under some conditions. It is not clear, however, what is the optimal mechanism for an exclusive-display auction, and future research can explore this.

Second, we find that search engine’s revenue and social welfare may increase or decrease in exclusive-display auctions, depending on equilibrium selection. Which equilibrium is more likely to happen, if bidding for exclusivity is allowed, needs further investigation. The equilibrium outcome may vary across different keywords as well. We showed that exclusive-display auctions perform better under ex-post Nash equilibrium concept. In other words,
if advertisers’ valuation for a given keyword is stable, and they tend to learn each others’ valuation over time as argued in Edelman et al. (2007), ex-post Nash equilibrium is more appropriate, and hence, exclusive-display auctions are likely to perform better. However, if advertisers frequently change their bids for a given keyword and the structure of the market makes it hard for competitors to learn each other’s strategies, using Bayesian Nash equilibrium may be more appropriate. We leave further investigation of equilibrium selection for future research.

Third, we make the assumption that every advertiser values exclusive display at least as much as multiple display, which is a very reasonable assumption. However, these valuations are assumed to be independent across competitors to keep the model simple. Explicitly modeling the effect of one advertiser on another advertiser is an interesting direction for future work. For example, a luxury car manufacturer such as Lexus may want to be listed exclusively if the competitive advertiser is another luxury car manufacturer such as Acura, but may care less about being listed next to a lower-quality manufacturer such as Kia. In other words, in the spirit of Jerath et al. (2011), the competitive environment of a firm may significantly influence its valuation for exclusive display and therefore its bidding strategy. Desai et al. (2010) study such effects in the context of a one-dimensional multiple-display auction. Future work can explicitly model these phenomena with the exclusive-display option also available to advertisers.

Finally, allowing each bidder to submit bids for multiple and exclusive display is simply one way to make the currently-prevailing auction format more expressive. However, there may be numerous other formats in which advertisers can reveal their preferences in more detail. For example, Muthukrishnan (2009) suggests a different two-dimensional auction in which each advertiser submits a per-click bid and declares the maximum number of other advertisers it wants to be displayed with. Future research can work towards a general theory of “expressive ad auctions.” This theory should also consider practical limitations such as ease of bidding by advertisers and the real-time calculation and implementation of auction outcomes by the search engine.

Credits: The results in this chapter are joint work with Kinshuk Jerath.
Chapter 5

Conclusion

Sponsored search advertising is new, is changing very fast and is rapidly growing. Since it is new, search engines need to constantly refine their pricing mechanisms and to introduce new dimensions to capture advertisers preferences more efficiently. Since it is changing very fast, businesses need to constantly work on and improve their advertising strategies. Finally, since it is rapidly growing, businesses have analyze implications of sponsored search on their marketing mix.

In my dissertation I study various aspects of sponsored search advertising. In the first essay, I examine the competition between firms. I show how competing for clicks, in sponsored search advertising, can affect businesses’ marketing decisions. In the second essay, I focus on the game between a search engine and firms. While firms constantly try to attain more clicks at lower price by gaming the system, search engine has to design auction rules in a way to prevent inefficient outcomes. Finally, in the third essay, I study competition between search engines which leads to innovation in sponsored search. I analyze the advantages and disadvantages of a new auction that is currently being investigated by search engines.

Changes, made by search engines, in sponsored search has been and will be a fruitful avenue for marketing researchers. Search engines have to decide what changes to make, and businesses have to understand the implications of these changes on their marketing decisions. The work by Zhu and Wilbur (2011), Amaldoss et al. (2011) and the third chapter of my thesis are examples of this trend of research.

Equilibrium analysis of sponsored search advertising has been an active area of research. Characterizing and better understanding equilibria is indeed very important for designing new mechanisms and analyzing their implications. The work by Edelman et al. (2007), Varian (2007), Athey and Nikipelov (2010) and the second chapter of this dissertation are examples of this line of research.

Finally, as sponsored search advertising grows and becomes more important, its interactions with other marketing mix variables may change. The impact on marketing decisions, which once could be ignored because sponsored search was too small, need to be analyzed now. The work by Desai et al. (2010), Jerath et al. (2011) and the first chapter of my thesis are examples of this trend of research.
Chapter 6

Appendix

6.1 Competitive Poaching in Sponsored Search Advertising and Strategic Impact on Traditional Advertising

6.1.1 Derivations for the Asymmetric Firms Case

We use the following terminology for brevity. When describing equilibria, we assume that the first firm is the weak firm, and the second firm is the strong firm. For example, by (Poach, Traditional) equilibrium we mean an equilibrium in which the weak firm poaches and the strong firm uses Traditional. We will also see two types of mixed equilibria. In the first mixed equilibrium, which we call the Weak-Poach-mixed equilibrium, the weak firm mixes between Poach and Own, and the strong firm mixes between Traditional and Own. In the second mixed equilibrium, which we call the Strong-Poach-mixed equilibrium, the weak firm mixes between Traditional and Own, and the strong firm mixes between Poach and Own.

Because of the existence of multiple equilibria in our setting, we define the weak dominance concept to compare sets of equilibria. We say that equilibrium set $S_1$ weakly dominates equilibrium set $S_2$, from Player $P$’s perspective, if for every equilibrium $e_1 \in S_1$ and $e_2 \in S_2$, Player $P$’s profit in $e_1$ is greater than or equal to her profit in $e_2$, with the inequality being strict for at least one pair $(e_1, e_2)$. This definition is particularly useful when comparing the revenue of the search engine with and without the presence of poaching.

We start the analysis by assuming a low level of asymmetry between the firms’ advertising budgets. Then, we show how the results are generalized for higher levels of asymmetry.

Low Level of Asymmetry

To aid the exposition of the derivation, we use Figure 6.1, which plots the firms’ and the search engine’s revenues for $\alpha = 0.5$ and $B = 1.5$ (which is a low asymmetry case). In Figure 6.1(a), we see the existence of multiple equilibria for $R > 0.60$. For $R \geq 0.6$, since $\Pi_{W,T}^W > \Pi_{O,O}^W$, the weak firm may poach on the strong firm’s keyword. Similarly, for $R > 1.58$ since $\Pi_{S,T}^S > \Pi_{S,O}^S$, the strong firm may poach on weak firm’s keyword. In general, let $R^W$
be the threshold value of $R$ for which $\Pi^{P,T}_W > \Pi^{O,O}_W$ if $R > R^W$. Similarly, let $R^S$ be the threshold value of $R$ for which $\Pi^{P,T}_S > \Pi^{O,O}_S$ if $R > R^S$. Using elementary calculus, it can be proved that $R^W < R^S$. In other words, the weak firm starts poaching for lower values of $R$. In the example of Figure 6.1, $R^W = 0.6$ and $R^S = 1.58$.

When $R < R^W$, the unique equilibrium is the (Own, Own). When $R$ is between $R^W$ and $R^S$, there are three equilibria: (Own, Own), (Poach, Traditional) and mixed. Note that in (Poach, Traditional) equilibrium, weak firm poaches and strong firm uses Traditional. Similarly, in mixed equilibrium, weak firm mixes between Poach and Own, and strong firm mixes between Traditional and Own. When $R$ is larger than $R^S$, there are five equilibria: (Own, Own), (Poach, Traditional), (Traditional, Poach) and two mixed equilibria. In the first mixed equilibrium, the weak firm mixes between Traditional and Own, and the strong firm mixes between Traditional and Own. However, in the second mixed equilibrium, the weak firm mixes between Traditional and Own, and the strong firm mixes between Traditional and Own. For brevity, throughout the rest of this section, we refer to the mixed equilibrium in which the weak firm mixes between Poach and Own, and the strong firm mixes between Traditional and Own as the Weak-Poach-mixed equilibrium, and to the other one, as the Strong-Poach-mixed equilibrium.

Figure 6.1(c) shows the revenue of the search engine for different equilibria as functions of reserve price $R$. In (Own, Own) equilibrium the search engine’s revenue is $1 + B – T_s^O – T_w^O$. In (Poach, Traditional) equilibrium (for $R > R^W$), the search engine’s revenue is $1 + B – T_s^T$. And, in (Traditional, Poach) equilibrium (for $R > R^S$), the search engine’s revenue is $1 + B – T_s^T$. In the Weak-Poach-mixed equilibrium, the search engine’s revenue is $1 + B – (1 – p_s^* )T_w^O – p_s^* T_s^T – (1 – p_s^* )T_s^O$, where $p_s^*$ and $p_s^*$ represent the probability of poaching of weak firm and Traditional of strong firm, respectively. Similarly, in the Strong-Poach-mixed equilibrium, the search engine’s revenue is $1 + B – (1 – p_s^* )T_w^O – p_w^* T_s^T – (1 – p_w^* )T_w^O$, where $p_w^*$ and $p_w^*$ represent the probability of Traditional of weak firm and poaching of strong firm, respectively. Note that the probabilities $p_s^*$, $p_w^*$, $p_s^*$ and $p_w^*$ can be calculated analytically, using the equilibrium conditions, as follows:

$$p_s^* \Pi^{P,T}_W + (1 – p_s^* )\Pi^{P,O}_W = p_s^* \Pi^{O,T}_W + (1 – p_s^* )\Pi^{O,O}_W \Rightarrow p_s^* = \frac{\Pi^{P,O}_W – \Pi^{O,O}_W}{\Pi^{P,O}_W + \Pi^{O,T}_W – \Pi^{P,T}_W – \Pi^{O,O}_W}$$

$$p_s^* \Pi^{T,P}_W + (1 – p_s^* )\Pi^{T,O}_W = p_s^* \Pi^{O,P}_W + (1 – p_s^* )\Pi^{O,O}_W \Rightarrow p_s^* = \frac{\Pi^{T,O}_W – \Pi^{O,O}_W}{\Pi^{T,O}_W + \Pi^{O,P}_W – \Pi^{T,P}_W – \Pi^{O,O}_W}$$

$$p_w^* \Pi^{T,P}_S + (1 – p_w^* )\Pi^{T,O}_S = p_w^* \Pi^{O,P}_S + (1 – p_w^* )\Pi^{O,O}_S \Rightarrow p_w^* = \frac{\Pi^{T,O}_S – \Pi^{O,O}_S}{\Pi^{T,O}_S + \Pi^{O,P}_S – \Pi^{T,P}_S – \Pi^{O,O}_S}$$

$$p_w^* \Pi^{P,T}_S + (1 – p_w^* )\Pi^{P,O}_S = p_w^* \Pi^{O,T}_S + (1 – p_w^* )\Pi^{O,O}_S \Rightarrow p_w^* = \frac{\Pi^{P,O}_S – \Pi^{O,O}_S}{\Pi^{P,O}_S + \Pi^{O,T}_S – \Pi^{P,T}_S – \Pi^{O,O}_S}$$

$^1$ $R^W$ is the value of $R$ at which $\Pi^{P,T}_W = \Pi^{O,O}_W$. The solution is $R^W = \sqrt{(1+\alpha)^2(1+\beta) + B – \alpha – 1} / (1+\alpha + \beta)$.

$^2$ $R^S = \sqrt{(1+\alpha)^2(1+\beta) + B – \alpha – 1} / (1+\alpha + \beta)$.  

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Let $R^* = \frac{\sqrt{1+B}}{1+B-\sqrt{1+B}}$. Using the expressions derived for search engine’s revenue, we have that the revenue of (Poach, Traditional) equilibrium is the same as the revenue of (Own, Own) equilibrium for any value of $R$ larger than $R^*$. In other words, for $R \geq R^*$, we have $1 + B - T^O_S = 1 + B - T^O_W - T^O_S$. In Figure 6.1(c), we have $R^* = 1.07$, showing the point where the curve representing the poaching equilibrium joins the curve representing the non-poaching equilibrium. We can similarly define $R^{**} = \frac{\sqrt{B(1+B)}}{1+B-\sqrt{B(1+B)}}$ to be the threshold value of $R$ beyond which the search engine’s revenue of (Traditional, Poach) equilibrium is equal to the revenue of (Own, Own) equilibrium. In other words, for $R \geq R^{**}$, we have $1 + B - T^T_O = 1 + B - T^T_O - T^T_S$. In Figure 6.1(c), we have $R^{**} = 1.72$, indicating the point where the curve representing the (Traditional, Poach) equilibrium joins the curve representing (Own, Own) equilibrium. When $R > R^*$ and $R < R^S$, not allowing poaching weakly dominates allowing poaching from search engine’s perspective. In this region, the revenues of (Own, Own) equilibrium and (Poach, Traditional) equilibrium are the same, and larger than the revenue of the mixed equilibrium. Similarly, when $R > R^{**}$, not allowing poaching weakly dominates allowing poaching. In this region, (Poach, Traditional), (Traditional, Poach) and (Own, Own) equilibria have the same revenue for the search engine; but they are higher than the revenues of the two mixed equilibria.

Let $R^*_m$ be the value of $R$ at which $1 + B - T^O_S - T^O_W = 1 + B - (1-p^*_W)T^O_W - p^*_S T^O_S$. In other words, $R^*_m$ is the value of $R$ at which the revenue of the search engine in the Weak-Poach-mixed equilibrium is equal to the revenue of the search engine in (Own, Own) equilibrium. In Figure 6.1(c), we have $R^*_m = 0.98$. For $R > R^W$ and $R < R^*_m$, revenues of the search engine from the Weak-Poach-mixed equilibrium and from (Poach, Traditional) equilibrium are larger than the revenue from (Own, Own) equilibrium. In other words, for $R^W \leq R < R^*_m$, the set of equilibria in presence of poaching (when poaching is allowed) weakly dominates the set of equilibria without poaching (when poaching is not allowed), from search engine’s perspective.

To summarize, $R$ can be in one of the following intervals:

1. $[0, R^W)$: The unique equilibrium is (Own, Own).
2. $[R^W, R^*_m)$: There are three equilibria: (Own, Own), (Poach, Traditional) and the Weak-Poach-mixed. Allowing poaching weakly dominates not allowing poaching from search engine’s perspective.
3. $[R^*_m, R^*)$: There are three equilibria: (Own, Own), (Poach, Traditional) and the Weak-Poach-mixed. Search engine’s revenue of the mixed equilibrium is lower than (Own, Own), and revenue of (Poach, Traditional) equilibrium is higher than (Own, Own) equilibrium.
4. $[R^*, R^S)$: There are three equilibria: (Own, Own), (Poach, Traditional) and the Weak-Poach-mixed. Not allowing poaching weakly dominates allowing poaching from search engine’s perspective.
5. $[R^S, R^{**})$: There are five equilibria. Search engine’s revenue may be lower or higher in presence of poaching, depending on equilibrium selection.

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6. \([R^*, \frac{1}{\alpha}]\): There are five equilibria. Not allowing poaching weakly dominates allowing poaching from search engine’s perspective.

7. \((\frac{1}{\alpha}, \infty)\): Both firms spend all of their budget on traditional channel. Search engine’s revenue is 0.

In case of symmetric firms, \(R^W = R^S\). In other words, intervals 2, 3 and 4 do not exist.

We also find that the weak firm’s relative gain from poaching is larger than that of the strong firm. In other words, \(\frac{\Pi_{P,T}^W}{\Pi_{O,O}^W} \geq \frac{\Pi_{P,T}^S}{\Pi_{O,O}^S}\). Moreover, the weak firm’s incentive to poach increases with increasing budget asymmetry. In other words, \(\frac{\Pi_{P,T}^W}{\Pi_{O,O}^W}\) is an increasing function of \(B\). This is intuitively because the strong firm has a relatively large search volume; therefore, the poaching of the weak firm does not affect the sponsored search price significantly, and in turn, allows poaching at a relatively low price. In fact, if the firms are very asymmetric, the incentive to poach is so high that \((\text{Poach, Traditional})\) is the only equilibrium of the game.

Next, we will discuss the equilibria of the game at medium and high levels of asymmetry.

**Medium and High Level of Asymmetry**

In this section, we study the effect of degree of budget asymmetry, on the results presented in the previous section. As we will show, the results are qualitatively similar. However, the degree of budget asymmetry has interesting effects on the size and location of the intervals.

The first interesting observation is that \(R^S\) is increasing in \(B\). In other words, as budget asymmetry increases the intervals in which the strong firm poaches on weak firm’s keyword (intervals 5 and 6) shrink. If \(B\) is large enough, \(R^S\) becomes larger than \(1/\alpha\). In other words, if one firm is enough larger than the other firm, the strong firm does not poach on the weak firm’s keyword under any condition.

Reverse of this effect exists for \(R^W\). As \(B\) increases, \(R^W\) decreases. If \(B\) is large enough, \(R^W\) becomes 0. In other words, if one firm is enough larger than the other firm, weak firm poaching on strong firm’s keyword is always an equilibrium. These changes in interval thresholds are consistent with Proposition 2.4.1 that says weak firm’s incentive to poach increase and strong firm’s incentive to poach decrease as budget asymmetry increases.

Mathematically speaking, if \(B \geq \frac{1}{\alpha}\) then \(R^S > \frac{1}{\alpha}\). Under this condition, strong firm does not poach on weak firm’s keyword. Furthermore, \(B \geq \frac{3+3\alpha+\alpha^2}{1+\alpha}\) implies \(R^W = 0\). Under
this condition, weak firm poaching on strong firm’s keyword is always an equilibrium. We define the values of $B$ where $B < \max(\frac{1}{\alpha}, \frac{3+3\alpha+\alpha^2}{1+\alpha})$ as low level of budget asymmetry. For such values of $B$, the results are what we discussed in the previous section. However, when $B \geq \max(\frac{1}{\alpha}, \frac{3+3\alpha+\alpha^2}{1+\alpha})$ we have medium or high level of asymmetry.

For medium level of asymmetry, $R$ can be in one of the following intervals.

1. $[0, R^*_m)$: There are three equilibria: (Own, Own), (Poach, Traditional) and Weak-Poach-mixed. Allowing poaching weakly dominates not allowing poaching from search engine’s perspective.

2. $[R^*_m, R^*)$: There are three equilibria: (Own, Own), (Poach, Traditional) and Weak-Poach-mixed. Search engine’s revenue of the mixed equilibrium is lower than (Own, Own), and revenue of (Poach, Traditional) equilibrium is higher than (Own, Own) equilibrium.

3. $[R^*, \frac{1}{\alpha}]$: There are three equilibria: (Own, Own), (Poach, Traditional) and Weak-Poach-mixed. Not allowing poaching weakly dominates allowing poaching from search engine’s perspective.

4. $(\frac{1}{\alpha}, \infty)$: Both firms spend all of their budget on traditional channel. Search engine’s revenue is 0.

Note that $R^* = \frac{\sqrt{1+B} - \sqrt{1+\frac{1}{\alpha}B}}{1+\frac{1}{\alpha}B - \sqrt{1+\frac{1}{\alpha}B}}$, is also a decreasing function of $B$. In other words, the first two intervals shrink and the third interval grows as $B$ increases.

A condition that could not exist for low level of asymmetry and could occur for high level of asymmetry is $\Pi_{P,O} > \Pi_{O,O}$. In other words, if the firms are asymmetric enough, even if the strong firm uses Own strategy, the weak firm prefers to poach. In this situation, (Own, Own) cannot be an equilibrium anymore. Using simple calculus, we see that this condition is satisfied if $B \geq 1 + \alpha$ and $R < \frac{-1-a+B}{1+a+aB}$. Define $\overline{R} = \frac{-1-a+B}{1+a+aB}$. Note that $\overline{R}$ converges to $\frac{1}{\alpha}$ as $B$ increases. This means that for large enough values of $B$, the only equilibrium is when weak firm poaches on strong firm’s keyword for almost all values of $R$ (except, of course for $R > \frac{1}{\alpha}$ where no firm uses sponsored search advertising at all). In summary, for high level of asymmetry, $R$ can be in one of the following intervals.

1. $[0, \overline{R})$: There is one equilibrium: (Poach, Traditional). Allowing poaching has the same revenue as not allowing poaching for the search engine.

2. $[\overline{R}, \frac{1}{\alpha}]$: There are three equilibria: (Own, Own), (Poach, Traditional) and Weak-Poach-mixed. Not allowing poaching weakly dominates allowing poaching from search engine’s perspective.

3. $(\frac{1}{\alpha}, \infty)$: Both firms spend all of their budget on traditional channel. Search engine’s revenue is 0.

As mentioned, the second interval shrinks and the first interval grows as $B$ increases. Finally, we should mention that the transition from the interval structure in medium level of asymmetry to the interval structure in high level of asymmetry is through the growth of $\overline{R}$. Depending on how large $\overline{R}$ is, the $[0, \overline{R})$ interval of high asymmetry case can override the fist interval, or the first and the second intervals of the medium asymmetry case.
6.1.2 Category Keyword

We extend our model and assume that there exists a category keyword which attracts customers from the traditional ads of both firms. We categorize the customers into three categories as follows: (1) The customer buys the product directly after seeing the traditional ad, or searches for the product keyword after seeing the traditional ad, but ignores the sponsored search results and eventually converts to the advertised product. (2) The customer searches for the product keyword after seeing the traditional ad, and converts to the product advertised in the sponsored search result; in the case that no product is advertised in sponsored search, the customer will not convert. (3) The customer searches for the category keyword after seeing the traditional ad, and converts to the product advertised in the sponsored search result; in the case that no product is advertised in sponsored search, the customer will not convert. We assume that the “scaled probability” that a customer is in Category 1 is $\alpha$, in Category 2 is 1, and in Category 3 is $\beta$. Therefore, if a firm spends $x$ in traditional advertising, there will be $\alpha x$ customers in Category 1, $x$ customers in Category 2, and $\beta x$ customers in Category 3. We now derive the expressions for the revenues of the firms under different strategies.

Definitions of Strategies

We rederive the budget allocations for strong and weak firms based on the core idea that when a firm is following a strategy of focusing on one of the three types of advertising, the allocation is such that the firm spends maximum amount of budget on that form of advertising but still uses an undominated strategy.

Let $C^J$ be the amount spent on the category keyword in sponsored search in strategy $J$, where $J \in \{N, P, D\}$; all other notation is carried over from the basic model.

**Own Strategy:** First, assume that there is only one firm in the market. If the firm spends $x$ in traditional advertising, to win the customers of the product keyword she has to spend at least $xR$ in sponsored search of the product keyword, and $\beta x R$ in sponsored search of the category keyword, where $R$ is the reserve price of sponsored search auction set by the search engine. The optimal amount of money to be spent on traditional advertising in this case is $T^O = B \frac{1}{1 + R(1 + \beta)}$. Consequently, since in Own strategy, the firm does not advertise on the keyword of the other firm (i.e., $S^O_2 = 0$), the amount of money that he spends on sponsored search is $S^O_1 + C^O = B - T^O = B \frac{R(1 + \beta)}{1 + R(1 + \beta)}$. Since the the number of queries to the product keyword and the category keyword are proportional to 1 and $\beta$, by Theorem 6.1.7 we have $S^O_1 \beta = C^O$, which gives $S^O_1 = B R \frac{1}{1 + R(1 + \beta)}$ and $C^O = B R \beta \frac{1}{1 + R(1 + \beta)}$.

**Poaching Strategy:** The poaching firm’s spending on its own keyword and on traditional advertising is zero, i.e., $T^P = S^P_1 = 0$. Using Theorem 6.1.7 the poaching firm’s spending on the competitor’s keyword is $S^P_2 = B / (1 + \beta)$, and on the category keyword is $C^P = \beta B / (1 + \beta)$.

**Traditional Strategy:** In the Traditional strategy, the firm assumes that the other firm poaches; given this assumption, the firm’s revenue is $\alpha T + (B - T) / R$ if $2B - T < T(1 + \beta) R$, and is $\alpha T + T(1 + \beta) \frac{B - T}{2B - T}$ otherwise. Assuming $\alpha < (1 + \beta) / R$, the optimal solution to this problem is $T^T = B(2 - \sqrt{2(1 + \beta) / (1 + \alpha + \beta)})$ if $\frac{2}{R(1 + \beta) + 1} \geq 2 - \sqrt{2(1 + \beta) / (1 + \alpha + \beta)}$, and $T^T = \frac{2B}{(1 + \beta) R + 1}$ otherwise. Since $S^T_1 + C^T = B - T^T$, by Theorem 6.1.7, $S^T_1 = \frac{1}{1 + \beta} (B - T^T)$.
Revenue Analysis

**Both Firms Own:** If both firms choose Own strategy, the revenue of each firm is \( \Pi_{O,O} = T^O (1 + \alpha + \beta) \).

**Both Firms Traditional:** If both firms choose Traditional strategy, the revenue of each firm is \( \Pi_{T,T} = \alpha T^T + \min(T^T, S^T) + \min(\beta T^T, C^T) \).

**Both Firms Poaching:** The revenue of both firms in this case is of course zero, i.e., \( \Pi_{P,P} = 0 \).

**One Firm Poaching, One Firm Own:** In this case, the number of queries on the product keyword is \( T^O \) and on the category keyword is \( \beta T^O \). Therefore, the Own firm’s revenue is \( \Pi_{O,P} = \alpha T^O + T^O \frac{s^O}{s^T_1 + s^T_2} + \beta T^O \frac{C^O}{C^P + C^O} \) and the Poaching firm’s revenue is \( \Pi_{P,O} = T^O \frac{s^P}{s^T_1 + s^T_2} + \beta T^O \frac{C^P}{C^P + C^O} \).

**One Firm Traditional, One Firm Own:** In this case, the number of queries on the Traditional firm’s product is \( T^T \) and on the Own firm’s product is \( T^O \). Also, the number of queries on the category keyword is \( \beta (T^O + T^T) \). Hence, the Own firm’s revenue is \( \Pi_{O,T} = \alpha T^O + T^O + \min(\frac{C^O}{R}, (\frac{C^O}{C^P + C^O}) \beta (T^O + T^T)) \), and the Traditional firm’s revenue is \( \Pi_{T,O} = \alpha T^T + \min(T^T, \frac{s^T}{R}) + \min(\frac{C^T}{R}, (\frac{C^T}{C^P + C^T}) \beta (T^O + T^T)) \).

**One Firm Poaching, One Firm Traditional:** In this case, the price will be greater than or equal to \( R \) for category keyword and the product keyword; therefore, \( \Pi_{T,P} = \alpha T^T + T^T \frac{s^T}{s^T_1 + s^T_2} + \beta T^T \frac{C^T}{C^P + C^T} \) and \( \Pi_{P,T} = T^T \frac{s^P}{s^T_1 + s^T_2} + \beta T^T \frac{C^P}{C^P + C^T} \).

We use the above expressions to analyze the equilibrium of the two-person normal-form game as before. We find that the results and insights from the basic model in Section 2.3 (without category keyword) continue to hold.

6.1.3 Reputation Effects Analysis

Suppose that Firm \( i \) has some exogenous search volume \( V_i \) for its keyword, which is independent of how much it has recently spent on creating awareness for its product. This may be, for instance, because of the previous reputation that the firm holds. For simplicity, we assume that \( V_1 = V \) and \( V_2 = 0 \), i.e., \( V \) customers search the keyword of the “strong” firm (denoted by subscript \( S \)) without traditional advertising, while no customers search the keyword of the “weak” firm (denoted by subscript \( W \)) without traditional advertising. As before, we assume that spending \( x \) on awareness advertising creates search volume \( x \); hence, if the strong firm spends \( x \) on awareness advertising, the search volume for its keyword will be \( V + x \).

Definitions of Strategies

We rederive the budget allocations for strong and weak firms based on the core idea that when a firm is following a strategy of focusing on one of the three types of advertising, the allocation is such that the firm spends maximum amount of budget on that form of advertising but still uses an undominated strategy.
if \( B \geq VR \) the consumers do not know about the existence of the firms.

We consider a Hotelling line of length 1, with consumers distributed uniformly on it and travel cost (misfit cost) along the line is \( t > 0 \) per unit distance traveled by a consumer. The consumers do not know about the existence of the firms.

**Traditional Strategy:** For the weak firm, the Traditional strategy does not change. However, notice that when the strong firm poaches, Traditional is not necessarily the best response from the weak firm as it may want to poach too. If \( \frac{2B}{R+1} \geq 2 - \sqrt{2/(1+\alpha)} \), \( T^T_W = B(2 - \sqrt{2/(1+\alpha)}) \); otherwise, \( T^T_W = 2B/(R+1) \). For the strong firm, recall that in Traditional strategy the firm assumes that the other firm poaches. Given this assumption, the firm’s revenue is \( \alpha T + (B - T)/R \) if \( 2B - T < (T + V)R \), and is \( \alpha T + (T + V)B/(2B - T) \) if \( 2B - T \geq (T + V)R \). Therefore, if \( \frac{2B - VR}{R+1} \geq 2 - \sqrt{2B/(1+\alpha)} \), then \( T^T_S = B(2 - \sqrt{2B/(1+\alpha)}) \); otherwise, if \( 2B \geq VR \), \( T^T_S = 2B - VR \); otherwise, \( T^T_S = 0 \).

**Poaching Strategy:** By definition, \( T^P_W = T^P_S = 0 \).

**Revenue Analysis**

**Own Strategy:** For the weak firm, as long as it is not poaching, \( V \) does not have an impact. Therefore, \( \Pi^O_W = \Pi^O_T = T^O_W(1 + \alpha) \) and \( \Pi^O_P = T^O_W(\alpha + \frac{B - T^O_W}{2B - T^O_W}) \). For the strong firm, if \( B \geq VR \), \( \Pi^O_S = \Pi^O_T = T^O_S(1 + \alpha) + V \); otherwise, \( \Pi^O_S = \Pi^O_T = B/R \). Similarly, if \( B \geq VR \), \( \Pi^O_P = T^O_S \alpha + \frac{B - T^O_S}{2B - T^O_S} (V + T^S_S) \); otherwise, \( T^O_S = 0 \) and hence, if \( 2B \geq VR \), \( \Pi^O_P = \frac{B}{2B} V = V/2 \); otherwise \( \Pi^O_S = B/R \).

**Traditional Strategy:** Nothing changes for the weaker firm, which implies \( \Pi^T_W = \alpha T^T_W + T^T_W \frac{B - T^T_W}{2B - T^T_W} \) and \( \Pi^T_W = \Pi^T_T = \alpha T^T_W + \min(T^T_W, \frac{B - T^T_W}{R}) \). For the strong firm, if \( 2B - T^T_S \geq (V + T^T_S)R \), \( \Pi^T_S = T^T_S \alpha + \frac{B - T^T_S}{2B - T^T_S} (T^T_S + V) \); otherwise, as in the previous case, \( \Pi^T_S = T^T_S \alpha + \frac{B - T^T_S}{R} \). Similarly, if \( 2B - T^T_S \geq (V + T^T_S)R \), \( \Pi^T_S = \Pi^T_T = T^T_S (1 + \alpha) + V \); otherwise, \( \Pi^T_S = \Pi^T_T = T^T_S \alpha + \frac{B - T^T_S}{R} \).

**Poaching Strategy:** If the stronger firm poaches, value of \( V \) does not affect its utility. Therefore, \( \Pi^P_S = 0 \), \( \Pi^P_W = T^O_W(\frac{B}{2B - T^O_W}) \) and \( \Pi^P_T = T^O_W(\frac{B}{2B - T^O_W}) \). If the weaker firm poaches, if \( B \geq VR \), \( \Pi^P_W = V \); otherwise, \( \Pi^P_W = B/R \). Similarly, if \( 2B - T^O_S \geq (V + T^O_S)R \), \( \Pi^P_W = \frac{B}{2B - T^O_S} (V + T^O_S) \); otherwise, \( \Pi^P_W = B/R \). Finally, if \( 2B - T^T_S \geq (V + T^T_S)R \), \( \Pi^P_W = \frac{B}{2B - T^T_S} (V + T^T_S) \); otherwise, \( \Pi^P_T = B/R \).

We use the above expressions to analyze the equilibrium of the two-person normal-form game as before.

### 6.1.4 Consumers’ Purchase Model and Price Competition Analysis

We consider a Hotelling line of length 1, with consumers distributed uniformly on it and each firm located at one end of the line. We assume that the valuation of each consumer for either firm’s product is \( V = 1 \), and travel cost (misfit cost) along the line is \( t > 0 \) per unit distance traveled by a consumer. The consumers do not know about the existence of the firms.
initially. A firm can make consumers aware of its product through traditional advertising. More specifically, if Firm $i$ spends $T_i$ on traditional advertising, $(1 + \alpha)T_i$ consumers become aware of Firm $i$’s product, and we assume that these consumers are uniformly distributed on the Hotelling line. After being exposed to traditional advertising, some consumers search the firm’s keyword on a search engine, in response to which they may see this firm’s ad or the competing firm’s ad. Some of the consumers who become aware of both firms (through one firm’s traditional ad and the other firm’s sponsored ad) compare prices before purchasing, which leads to price competition.

The consumers who eventually purchase the product from Firm $i$ could be in one of the following categories ($j = 3 - i$ is the index of Firm $i$’s competitor):

1. Exposed to traditional advertising of Firm $i$, not influenced by sponsored search advertising and purchase from Firm $i$;

2. Exposed to traditional advertising of Firm $i$, influenced by sponsored search advertising, see sponsored search advertising of Firm $i$ and purchase from Firm $i$;

3. Exposed to traditional advertising of Firm $j$, influenced by sponsored search advertising, see sponsored search advertising of Firm $i$, without comparing prices purchase from Firm $i$;

4. Exposed to traditional advertising of Firm $i$, influenced by sponsored search advertising, see sponsored search advertising of Firm $j$, compare prices and purchase from Firm $i$;

5. Exposed to traditional advertising of Firm $j$, influenced by sponsored search advertising, see sponsored search advertising of Firm $i$, compare prices and purchase from Firm $i$.

Consumers in Category 1 are not influenced by sponsored search. Consumers in Categories 2, 3, 4 and 5 are influenced by sponsored search. Price competition between firms is only due to Categories 4 and 5. This feature of the model implies that consumers who are poached, and therefore become aware of both firms, also compare prices across firms, which leads to price competition.

Let $C_{a,b}$ (where $a, b \in \{1, 2\}$) be the number of customers who are exposed to traditional advertising of Firm $i$ and sponsored search advertising of Firm $j$, where each $C_{a,b}$ is a function of the firms’ advertising budget allocations. There is a total of $C_{1,2} + C_{2,1}$ customers who are exposed to advertising (traditional or sponsored search) of both firms. We assume that $\chi$ fraction of them compare the prices of the two firms while $1 - \chi$ fraction purchase from the firm that is shown in sponsored search. From Firm $i$’s point of view, Categories 1, 2, 3, 4 and 5 have $\alpha T_i, C_{i,i}, (1 - \chi)C_{j,i}, \chi C_{i,j}$ and $\chi C_{j,i}$ consumers, respectively. The number of

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\[3\] We assume that the total consumer population is large enough that it is unlikely that a consumer is exposed to traditional advertising of both firms. Therefore, after traditional advertising, each consumer knows about at most one product.

\[4\] Note that those customers who are exposed to advertising of both firms but without comparing prices purchase from the firm that did traditional advertising are already counted in $\alpha T_i$ and are categorized in the Category 1.
consumers in each category depend on the firms’ advertising budget allocations. Using the formulation of Section 4.3, we have \( C_{i,j} = \min(T_i, \frac{P_j}{S_i} + P_i R_i) \), where \( T_i \), \( S_i \) and \( P_i \) represent how much Firm \( i \) spends on traditional advertising, sponsored search advertising of its own keyword, and poaching on competitor’s keyword, respectively.

For Categories 1, 2 and 3, in which consumers do not compare prices across firms, \((1 - p_i)/t\) of the consumers purchase. For Categories 4 and 5, in which consumers compare prices, \(1/2 + (p_j - p_i)/(2t)\) of the consumers purchase from Firm \( i \) (and the rest from Firm \( j \)).

Therefore, assuming that the marginal cost of production is zero, the profit of Firm \( i \) is:

\[
\Pi_i = p_i \left( (\alpha T_i + C_{i,i} + (1 - \chi)C_{j,i}) \frac{1 - p_i}{t} + \chi(C_{i,j} + C_{j,i}) (\frac{1}{2} + \frac{p_j - p_i}{2t}) \right).
\]

The parameter \( \chi \) captures the price competition between the firms due to poaching. Note that if \( \chi = 0 \), the model collapses to the model in Section 4.3 (and the optimal price is \( 1/2 \)).

We solve the above model numerically and confirm that the results presented in Sections 2.3 and 2.4 are robust under price competition. We see that symmetric firms may use different strategies in equilibrium with one firm focusing on traditional advertising and the other firm focusing on poaching. Moreover, as in Section 2.5, the search engine’s revenue is maximized with a medium level of penalty on poaching. A new interesting result from this model is that the poaching firm sets a lower price than the other firm. In this way, the poaching firm can maximize the effect of poaching on its competitor’s keyword and win more of the comparison shoppers. The firm that is being poached does not decrease the price as much because it is benefiting from the customers who are not aware of the product of the poaching firm (i.e., customers not influenced by sponsored search).

### 6.1.5 Strategy Space Discretization

In the model in Section 4.3, we discretize the game by restricting the strategy space of each firm to three strategies, namely, Own, Poaching and Traditional. Although we allow the firms to use mixed strategies, we do not allow them to split their budget among the channels as a pure strategy. The purpose of the discretization is to make the model easier to solve and understand. In contrast, we could leave the strategy space continuous, allowing each firm to allocate arbitrary portion of its budget to each of the channels. In this section, we show that our results are robust under continuous strategy space. In particular, we show that under certain conditions, the set of equilibria when the strategy space of the firms is continuous coincides with the set of equilibria when their strategy space is discrete.

Let game \( G \) be the discrete game between the two firms as defined in Section 4.3. Let game \( H \) be the continuous version of game \( G \). In other words, in game \( H \), each firm decides how to allocate its budget to different channels of advertising, and does not have to necessarily follow exactly one of the three Poaching, Own or Traditional strategies. We show that when the reserve price \( R \) is large enough, the poaching equilibrium of game \( G \)

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5Let \( x \) be a consumer’s distance from Firm \( i \), and \( p_i \) and \( p_j \) be the prices of Firms \( i \) and \( j \), respectively. If this consumer considers only Firm \( i \), she purchases if \( 1 - tx - p_i \geq 0 \). If she considers both firms, she purchases from Firm \( i \) if \( 1 - tx - p_i \geq 1 - t(1 - x) - p_j \). While solving, we consider boundary effects as needed.
is also an equilibrium of game $H$. Furthermore, if the relevance score $\gamma$ is small enough, the non-poaching equilibrium of $G$ is also an equilibrium of $H$. Finally, we prove that if $R$ is large enough, game $H$ has no equilibrium other than those in $G$. Therefore, the results obtained for discrete game $G$ throughout the paper also apply to continuous game $H$.

**Theorem 6.1.1.** For sufficiently large reserve price $R$ (when $R \geq \frac{2B-T}{T+1}$), the poaching equilibrium of the discrete game $G$ is also an equilibrium of the continuous game $H$.

**Proof.** Consider a poaching equilibrium of game $G$ where Firm 1 poaches on Firm 2’s keyword. Firm 2’s response, Traditional strategy, is calculated over the continuous strategy space and hence is a best response to poaching of Firm 1 in the continuous game $H$ as well, by definition. Firm 1’s utility from poaching all of its budget is $\frac{B}{R}$ because $R \geq \frac{2B-T}{T+1}$. If Firm 1 deviates and spends $x$ on poaching and $B-x$ on traditional channel and its own keyword, assuming that it splits $B-x$ optimally between traditional channel and its own keyword, its utility will be $\frac{x}{R} + (1+\alpha)\frac{B-x}{R+1}$. From Section 2.3 we know that $\alpha < \frac{1}{R}$. This proves that the deviation is dominated for any $x < B$. Therefore, poaching with all of the budget is best response to Traditional. Consequently, poaching equilibrium of discrete game $G$ is also an equilibrium of continuous game $H$. \hfill \Box

The theorem below states that in equilibrium, one firm spends all of its budget poaching on the other firm’s keyword.

**Theorem 6.1.2.** For sufficiently large reserve price $R$ (when $R \geq \frac{2B-T}{T+1}$), the poaching equilibrium of the continuous game $H$ is the only equilibrium of game $H$.

**Proof.** Suppose that in equilibrium Firm $i$ spends $t_i$, $s_i$ and $p_i$ on traditional channel, its own keyword and competitor’s keyword, respectively. Let $x_i = t_i + s_i$, and $y_i = p_i$. In other words, $x_i$ corresponds to the amount of money that a firm spends on its own channel, and $y_i$ corresponds to the amount of money that a firm spends on its competitor’s channel. We know that, not considering the opponent’s response, increasing $x_i$ does not affect the marginal utility of $y_i$ and vice versa. Therefore, by first order conditions, in equilibrium, the marginal utility of $x_i$ and $y_i$ must be equal unless $y_i = 0$ or $y_i = B$.

First, suppose that $y_i \neq 0$ and $y_i \neq B$ for both firms. Let $d_{x_1}$ and $d_{y_1}$ denote the marginal utility of $x_i$ and $y_i$. We have $d_{x_1} = d_{y_1}$ and $d_{x_2} = d_{y_2}$. Lemma 6.1.3 shows that $d_{y_1} > d_{x_2}$ and $d_{y_2} > d_{x_1}$. This proves that $d_{x_1} = d_{y_1} > d_{x_2} = d_{y_2} > d_{x_1}$ which is a contradiction. Therefore, we conclude that at least one of $y_i$s must be $B$ or 0.

Now, we prove that at least one of $y_i$s must be $B$. Suppose that $y_1 < B$ and $y_2 < B$. This implies that $d_{y_1} \leq d_{x_1}$ and $d_{y_2} \leq d_{x_2}$. Using the same argument as before, we get $d_{x_1} \geq d_{y_1} > d_{x_2} \geq d_{y_2} > d_{x_1}$ which is a contradiction. \hfill \Box

**Lemma 6.1.3.** For sufficiently large reserve price $R$ (when $R \geq \frac{x_i+y_i-T}{T+1}$), assuming equilibrium conditions, we have $d_{y_i} > d_{x_j}$ for $i \neq j$.

**Proof.** When $R$ is sufficiently large, using basic calculus, we get $d_{y_i} = \frac{1}{R}$ and $d_{x_j} = \frac{\alpha+1}{R+1}$. Since $R < \frac{1}{\alpha}$, we have $d_{y_i} > d_{x_j}$. \hfill \Box
Theorem 6.1.2 shows that the continuous game $H$ is slightly different from game $G$ as there is no non-poaching equilibrium in game $H$. However, Theorem 6.1.4 shows that if poaching is penalized, the non-poaching equilibrium of discrete game $G$ is also an equilibrium of game $H$.

**Theorem 6.1.4.** If the poaching is penalized by a multiplier $\gamma$, for sufficiently small $\gamma$ (where $\gamma \leq \frac{(1+\alpha)R}{1+R}$) non-poaching equilibrium of game $G$ is also an equilibrium of game $H$.

**Proof.** Consider the non-poaching equilibrium of $G$ in which each firm uses Own strategy. We know, by definition, that as long as the firms do not want to poach on each other’s keywords, Own strategy is the optimum way of splitting budget between traditional channel and own keyword on sponsored search. Consider a deviation where Firm 1 spends $x > 0$ poaching on Firm 2’s keyword while Firm 2 is playing Own strategy. Also, assume that Firm 1 splits the remaining $B - x$ optimally between between traditional channel and its own keyword. Firm 1’s utility after deviation is $\gamma T^O x B - T^O + x + (1 + \alpha) B^{1+1} R^{1+1}$ while before deviation it is $(1 + \alpha) B R^{1+1}$. Using elementary calculus we see that the deviation is beneficial if and only if $\gamma \geq \frac{(1+\alpha)R}{1+R}$. Therefore, if $\gamma \leq \frac{(1+\alpha)R}{1+R}$ then non-poaching equilibrium of game $G$ is also an equilibrium of game $H$. 

### 6.1.6 Proofs of Theorems Used As Intermediate Results

Suppose that a seller want to sell $n$ units of an item. The seller can sell the units one by one, each in a second price auction. We call this mechanism a sequential second price auction. This mechanism roughly describes how search engines sell their advertising slots. Whenever a consumer searches a keyword, the search engine runs a (generalized) second price auction to sell the advertising slot. The seller can instead sell the $n$ units using a market-clearing-price mechanism. In the market-clearing-price mechanism, the seller sets the highest price $p$ at which demand meets supply. The following theorem proves that the two mechanisms essentially lead to the same outcome.

**Theorem 6.1.5.** Suppose that $n$ identical items are sold in a sequential second price auction with reserve price $R$. Two bidders 1 and 2 with budgets $B_1$ and $B_2$ are participating in the auctions; each bidder wants to maximize the number of items that she wins. The outcome of any subgame perfect equilibrium of the game is equivalent to the outcome of market clearing price mechanism with reserve price $R$.

**Proof.** First suppose $[B_1/R] + [B_2/R] \geq n$, i.e., the market clearing price is at least $R$. Let $p$ be the market clearing price; i.e., $[B_1/p] + [B_2/p] = n$. Note that if the first player bids $p$ in all rounds, he can make sure that he wins at least $n - [B_2/p] = [B_1/p]$ items because his opponent has to pay $p$ for every item that he wins. Similarly, if the second player bids $p$ in all rounds, he can make sure that he wins at least $n - [B_1/p] = [B_2/p]$ items. Since, $[B_1/p] + [B_2/p] = n$, we see that player $i$ cannot win more than $[B_i/p]$ items, which means that he wins exactly $[B_i/p]$ items.

Now, consider the case where $[B_1/R] + [B_2/R] < n$. In this case, we know that if the largest bid in the auction is smaller than $R$, the item in that round will be left unallocated. Also, if the larger bid is at least $R$, but the smaller bid is less than $R$, the item will be
allocated, but at price \( R \) (instead of the second highest bid). Given this information, bidding anything below \( R \), in any round, is weakly dominated. Also, by bidding \( R \), bidder \( i \) can make sure that he wins at least \( \lfloor B_i / R \rfloor \) items. Since bidder \( i \) can never win more than \( \lfloor B_i / R \rfloor \) items, in any subgame perfect equilibrium, he wins exactly \( \lfloor B_i / R \rfloor \) items.

Note that the subgame perfect equilibrium of a sequential second price auction is not unique, and there are many different optimal actions that the players may take in each period. However, they all eventually lead to the same outcome described in Theorem 6.1.5. The result above is also robust to different variations to the model. For instance, if all of the customers arrive at once, or if the firms cannot change the bids for each customer, or if the search engine uses a first-price auction instead of a second-price auction, we get the same outcome. The result can also be extended to two slots per keyword instead of one slot under the condition \( B_1 / (B_1 + B_2) \leq c_1 / (c_1 + c_2) \), where \( c_1 \) and \( c_2 \) are the click-through rates of slots 1 and 2, respectively, with \( c_1 \geq c_2 > 0 \), and \( B_1 \geq B_2 > 0 \). Moreover, if the search engine allows one firm’s ads to be placed in more than one slot (e.g., a search for “Toyota” may return ads from “toyota.com” and “buyatoyota.com,” both of which actually belong to Toyota) then the above condition is not needed for the theorem to hold. The result is also robust to unequal valuations of the advertisers for clicks, as long as the valuations are high enough (given their budgets and the number of units being sold) to guarantee a unique market-clearing price. In the case of equal valuations, we assume that the search engine uses a rule that if bids are equal, it will choose the advertiser with the larger remaining budget as the winner. Search engines do not reveal the details of their mechanisms. However, they claim that their mechanisms try to keep the budgets of the advertisers non-zero until the end of the campaign, which is the essence of our assumption above. This practice is also called “bid throttling” in sponsored search advertising parlance. While the advertisers benefit from this because their campaigns keep running until the end of the month, their non-winning bids are being used against their competitors by the search engine. More details about the different aspects of the proof are available from the authors upon request.

**Lemma 6.1.6.** The function \( \frac{x}{C+x} \) is monotonically increasing and concave in \( x \), for any \( x \geq 0 \) and fixed \( C \geq 0 \).

**Proof.** The first derivative in \( x \) is \( \frac{C}{(C+x)^2} \) and the second derivative is \( -\frac{2C}{(C+x)^3} \). □

**Theorem 6.1.7.** Suppose that there are two investment options. The revenue of the first one has the functional form \( \alpha Q \frac{x}{\alpha C+x} \) if \( x \) is invested, while the revenue of the second one is \( \beta Q \frac{x}{\beta C+x} \). Then, the optimal way to split \( x \) between the two options is to invest \( \frac{\alpha x}{\alpha + \beta} \) in the first one and \( \frac{\beta x}{\alpha + \beta} \) in the second one.

**Proof.** The proof directly follows from Lemma 6.1.6 and first-order conditions. □
6.2 Budget Constraints in Sponsored Search Advertising

6.2.1 Proof of Proposition 1

First, note that \( x(\alpha^c, b) \) is weakly increasing in \( c \) since \( \alpha \) is nonincreasing, for \( c' \geq c \geq 0 \), we have \( \alpha^{c'}(y) = \alpha(y + c') \leq \alpha(y + c) = \alpha^c(y) \), and hence

\[
x(\alpha^c, b) = \int_0^b \frac{1}{\alpha^c(y)} dy \geq \int_0^b \frac{1}{\alpha^c(y)} dy = x(\alpha^c, b).
\]

Also, obviously \( x(\alpha^c, b) \) is strictly increasing in \( b \).

Now we can show that \( X(c, (b, v)) \) is strictly increasing in \( c \). Consider \( c' > c \geq 0 \); we have

\[
X(c, (b, v)) = \left( \sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c+s}, b_j - s)
\]

where \( j \) satisfies \( c \leq \sum_{i=1}^j b_i \) and \( c > \sum_{i=1}^{j-1} b_i \) (and \( s = \sum_{i=1}^j b_i - c \)). For \( c' > c \), we can have one of two cases: either the index \( j \) stays the same or \( j \) is larger.

If \( j \) is larger, then we have

\[
X(c', (b, v)) > \sum_{i=1}^{j} x(\alpha^{c'}, b_i)
\]

\[
\geq \left( \sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c'}, b_j) > X(c, (b, v))
\]

This is because \( x(\alpha^{c'}, b_i) \geq x(\alpha^c, b_i) \) for all \( i = 1, ..., j-1 \) and \( x(\alpha^{c'}, b_j) > x(\alpha^{c+s}, b_j - s) \) since \( c' > c + s \).

If \( j \) stays the same (i.e., if \( c' < c + s \)), then we have

\[
X(c', (b, v)) = \left( \sum_{i=1}^{j-1} x(\alpha^{c'}, b_i) \right) + x(\alpha^{c+s'}, b_j - s')
\]

\[
> \left( \sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c+s}, b_j - s) = X(c, (b, v))
\]

where \( s' = \sum_{i=1}^j b_i - c' < s \). This is because \( x(\alpha^{c'}, b_i) \geq x(\alpha^c, b_i) \) for all \( i = 1, ..., j-1 \) and \( x(\alpha^{c+s'}, b_j - s') > x(\alpha^{c+s}, b_j - s) \) since \( c' + s' = c + s \) and \( b_j - s' > b_j - s \).

Next we show that \( X(c, (b, v)) \) is continuous in \( c \). By definition, \( x(\alpha^c, b) \) is continuous in \( c \) and \( b \) (This is because \( x(\alpha^c, b) = \int_0^b \frac{1}{\alpha(y+c)} dy \) and is continuous in \( c \) and \( b \) even when \( \alpha \) is not a continuous function). Moreover

\[
X(c, (b, v)) = \left( \sum_{i=1}^{j-1} x(\alpha^c, b_i) \right) + x(\alpha^{c+s}, b_j - s).
\]

If \( c \) increases from \( c \) to \( c + \epsilon \), \( j \) changes only when \( s = 0 \). If \( s \neq 0 \), then \( X(c, (b, v)) \) is obviously continuous in \( c \) as all of the terms in the summation are continuous in \( c \). If \( s = 0 \),
then
\[ X(c + \varepsilon, (b, v)) = \left( \sum_{i=1}^{j} x(\alpha^{c+\varepsilon}, b_i) \right) + x(\alpha^{c+\varepsilon+s'}, b_{j+1} - s') \]
and this goes to \( X(c, (b, v)) \) as \( \varepsilon \) goes to zero. This is because \( \sum_{i=1}^{j} x(\alpha^{c+\varepsilon}, b_i) \to \sum_{i=1}^{j} x(\alpha^{c}, b_i) = X(c, (b, v)) \) and \( x(\alpha^{c+\varepsilon+s'}, b_{j+1} - s') \to 0 \) since \( s' \to b_{j+1} \).

Hence we conclude that \( X(c, (b, v)) \) is strictly increasing and continuous in \( c \).

We consider pricing rules that are not too high, in the sense that they will be able to sell all the items if all budgets are exhausted. Hence we assume that for \( B \equiv \sum_{i=1}^{n} b_i \), we have
\[ \alpha(B) \leq B. \]

With this assumption, we can easily conclude that \( X(B, (b, v)) \geq 1 \). This is because when \( c = 1 \), all bidders are full winners and their allocations satisfy
\[ x(\alpha^{B}, b_i) \geq \frac{b_i}{\sum_{i=1}^{n} b_i} \]
and hence
\[ X(B, (b, v)) = \sum_{i=1}^{n} x(\alpha^{B}, b_i) \geq 1. \]

Thus we conclude that there is a unique \( c^* \) such that \( X(c^*, (b, v)) = 1. \)

### 6.2.2 Proof of Proposition 2

Consider bidder \( i \) with announced type \((b_i, v_i)\).

- First we show that revenue is nondecreasing in budgets. Consider bidder \( i \) who decreases her budget to \( b_i^- < b_i \). We show that revenue cannot increase with this deviation.

  - If bidder \( i \) was originally a loser by announcing \((b_i, v_i)\), then she cannot become a winner or partial winner by deviating to \( b_i^- < b_i \). This is because by this deviation, the pricing function for everybody becomes better and winners pay less per unit. Therefore the revenue cannot increase.

  - Next, consider bidder \( i \) who is a partial winner by bidding \( b_i \). If bidder \( i \) deviates to \( b_i^- \) and becomes a loser, then the revenue has to decrease since the set of losers becomes larger with this deviation. If she deviates to \( b_i^- \) and remains a partial winner, since all winners’ pricing gets better, the revenue has to decrease. If she deviates to \( b_i^- \), she cannot become a full winner. If this were the case, the pricing function for every (full or partial) winner gets better, and then the total number of units allocated will be greater than one.

  - Lastly, consider bidder \( i \) who is originally a winner by announcing \((b_i, v_i)\). If she deviates to \( b_i^- \) and if she becomes a loser or a partial winner after the deviation, then the revenue clearly decreases. This is because the set of full winners before the deviation is a strict superset of the set of full winners after the deviation. Now consider the case where bidder \( i \) deviates to \( b_i^- \) and remains a winner. Let us...
denote $b_i - b_i^-$ by $\Delta$. Suppose that the initial cut point is $c$ and the new cut point after the deviation is $c'$. Let $\alpha$ be the $n$-piece step function defined by $(b, v)$. Note that the initial revenue is $c$ and the new revenue is $c'$. We will show that $c \geq c'$.

Since $i$ has understated her budget, there will be a shortage of demand and the pricing of all original winners will be better. Therefore, with this deviation, all original winners except $i$ will be allocated (weakly) more units of the object. Assume for a contradiction that $c' > c$. This means that there will be new winners who use an extra budget strictly greater than $\Delta$, say $\Delta'$. We now argue that the extra units allocated to these new winners have to be greater than the number of units bidder $i$ is giving up with the deviation. Extra units allocated to new winners are priced at the values starting from the new cut point $c + \Delta'$ (according to $(b, v)$) and the total budget used is $\Delta'$. The number of units bidder $i$ is giving up are priced at the values in the range of $c$ to $c + \Delta < c + \Delta'$ and the total budget used is $\Delta$. Since extra units are given with higher budget ($\Delta' > \Delta$) and lesser prices ($c + \Delta < c + \Delta'$) than the units given up, we conclude that with the assumption $c' > c$, the total number of units allocated has to be strictly greater than one, which is a contradiction.

We can present this argument more formally. Consider the case in which $\Delta$ is small enough so that the original partial winner $j$ remains a partial winner. All full winners $k \neq i$ with $k < j$ will be allocated more items since bidder $j$ will be using more of her budget after the deviation. Let us consider the difference between the total amounts allocated to bidders $i$ and $j$ before and after the deviation. Bidder $i$’s allocation is decreased by

$$A \equiv x(\alpha^c, b) - x(\alpha^{c+\Delta'}, b - \Delta)$$

since

$$x(\alpha^{c+\Delta'}, b - \Delta) > x(\alpha^{c+\Delta'}, b - \Delta').$$

We have

$$A < x(\alpha^c, b) - x(\alpha^{c+\Delta'}, b - \Delta') = x(\alpha^c, \Delta').$$

On the other hand, bidder $j$’s allocation is increased by

$$B \equiv x(\alpha^{c+s}, b_j - s + \Delta') - x(\alpha^{c+s}, b_j - s) = x(\alpha^{c+b_j}, \Delta').$$

Since

$$x(\alpha^{c+b_j}, \Delta') \geq x(\alpha^c, \Delta'),$$

we conclude that $B > A$. The argument for the case when the deviation results in a change of the partial winner is very similar but not illuminating. Thus, the total number of units allocated has to increase after the deviation.
Now, we show that revenue is increasing in values. Consider bidder \( i \) who increases her value to \( v_i^+ > v_i \). We show that revenue cannot decrease with this deviation.

- First, if bidder \( i \) is a winner by bidding \((b_i, v_i)\) and she deviates to \( v_i^+ > v_i \), then she remains a winner after the deviation, and the revenue does not change. This is because the allocation and pricing rule of Vickrey with Budgets is invariant to full winners’ values (so long as they remain full winners).

- Second, consider a bidder \( i \) who is a loser by bidding \((b_i, v_i)\) and deviates to \( v_i^+ > v_i \). If she remains a loser after the deviation, since the pricing function for winners gets worse, the revenue has to increase. Let us now consider the deviation which makes bidder \( i \) a partial winner. If the original partial winner becomes a full winner after the deviation \((v_i^+ < v_j \text{ where } j \text{ is the original partial winner})\), the revenue obviously increases with the deviation, since the cut-point has increased.

Let us consider the case in which \( v_i^+ > v_j \): bidder \( i \) becomes a partial winner and bidder \( j \) becomes a loser after the deviation. Assume for contradiction that the revenue decreases with the deviation. If this is the case, it can be seen that the pricing function for all winners becomes worse after the deviation (the total unspent budget of price setters with \( v_k \geq v_i \) becomes greater and some of the values increase). Hence all full winners will be allocated less units of items after the deviation. This implies that the number of units allocated to \( i \) after the deviation has to be greater than the number of units allocated to \( j \) before the deviation. But again, the pricing function for \( i \) after the deviation is worse than the pricing function for \( j \) before the deviation. For \( i \) to be allocated more, her budget spent after the deviation has to be greater than \( j \)'s budget spent before the deviation, which is a contradiction.

Suppose bidder \( i \) is currently a loser and deviates to \( v_i^+ \) and becomes a full winner. We can split this into two deviations. First, \( i \) deviates to \( v_i^{++} > v_j \) and becomes a partial winner (which increases the revenue), then she deviates to \( v_i^+ \) and becomes a full winner which will next be shown to increase the revenue.

- Lastly, consider bidder \( i \) who is a partial winner by bidding \((b_i, v_i)\). It is obvious that she cannot become a loser after deviating to \( v_i^+ \). If she deviates to \( v_i^+ \) and remains a partial winner, then the pricing function for all winners get worse, hence the revenue has to increase. If she deviates to \( v_i^+ \) and becomes a full winner, then we argue that revenue has to increase.

Consider the case where bidder \( i \) is currently the partial winner, and she deviates to \( v_i^+ > v_{i-1} \) (where bidder \( i - 1 \) has the next highest value after bidder \( i \)) so that \( i - 1 \) is the new partial winner and \( i \) is a full winner. Denote the original unused budget of bidder \( i \) by \( s'_i \) and after deviation, the unused budget of bidder \( i - 1 \) by \( s'_{i-1} \). It suffices to show that \( s'_i \geq s'_{i-1} \). Assume for contradiction that \( s'_{i-1} > s'_i \). First it is easy to see that the pricing function for all winners other than \( i \) or \( i - 1 \) gets worse, therefore they will be allocated (weakly) less number of items. As in the previous discussion, we show that the total number of units allocated
to bidder \(i\) and \(i - 1\) has to (strictly) decrease after the deviation, which gives us the desired contradiction. Bidder \(i - 1\)'s allocation is decreased by

\[
x(\alpha^c, b_{i-1}) - x\left(\alpha^{c+s_i'}, b_{i-1} - s_{i-1}'\right)
\]

which is strictly greater than

\[
x(\alpha^c, s_i')
\]

Bidder \(i\)'s allocation is increased by at most

\[
x\left(\alpha^{c+s_i'-s_{i-1}'}, b_i\right) - x\left(\alpha^{c+s_i'}, b_i - s_i'\right)
\]

which is smaller than

\[
x\left(\alpha^{c+s_i'-s_{i-1}'}, s_i'\right)
\]

Since \(c + s_i' - s_{i-1}' < c\), we conclude that the total number of units allocated to players has to be strictly less than one, leading to a contradiction.

6.3 Exclusive Display in Sponsored Search Advertising

6.3.1 The \(VCG_{2D}\) Auction

The \(VCG_{2D}\) auction is an extension of the one-dimensional \(VCG\) auction to two dimensions.

**The \(VCG_{2D}\) auction** The \(VCG_{2D}\) mechanism compares \(b^E_{\text{max}}\) and \(\sum_{i=1}^k \theta_i b^N_i\).

- If \(b^E_{\text{max}} \geq \sum_{i=1}^k \theta_i b^N_i\), VCG allocates the page to only one advertiser, namely max, and charges him either the sum of the \(k\) highest \(\theta_i b^N_i\)'s (excluding himself) or the second highest S-bid, whichever is larger, *i.e.*, the winner’s payment is

\[
\max(b^E_{\text{max}}^2, \sum_{i=1}^{\text{max}-1} \theta_i b^N_i + \sum_{i=\text{max}}^k \theta_i b^N_i).
\]

- If \(b^E_{\text{max}} < \sum_{i=1}^k \theta_i b^N_i\), then VCG allocation is M, but the expression for the payments is more complicated. When advertiser \(i\) is removed, the efficient reallocation can be either S or M. If it is S, the winner is max\(-i\), and hence, the increase in the sum of the values of all advertisers other than \(i\) is \(b^E_{\text{max}-i} - \sum_{j \neq i}^k \theta_j b^N_j\). If the efficient reallocation is M, all advertisers below \(i\) will move one slot up, therefore, the sum of their values increases by \(\sum_{j=i}^k (\theta_j - \theta_{j+1}) b^N_{j+1}\). Therefore, the \(i\)-th advertiser’s payment, \(\theta_i p_i\), is (for \(i \leq k\))

\[
\max\left(\sum_{j=i}^k (\theta_j - \theta_{j+1}) b^N_{j+1}, b^E_{\text{max}-i} - \sum_{j=1}^k \theta_j b^N_j + \theta_i b^E_i\right).
\]

\(VCG_{2D}\) has some notable characteristics. First, it is a truthful mechanism (*i.e.*, the best strategy for the advertisers is to be truthful) which makes it a stable mechanism as well.
Second, it maximizes social welfare because the outcome is chosen as S or M and slots are subsequently allocated to the advertisers based on who values them more. For the simplified model in Section 4.4 we obtain the following result for \( VCG_{2D} \).

**Lemma 6.3.1.** If \( c > a + bp \), the revenue of \( VCG_{2D} \) is \( a + bp \), otherwise, the revenue is \( \max(c, a) + \max(c, b) - a - bp \). Its social welfare is \( \max(c, a + bp) \).

*Proof.* If \( c \geq a + bp \), the outcome is S; by the \( VCG_{2D} \) payment rule, the price is \( a + bp \). If \( c < a + bp \), the outcome is M; by the \( VCG_{2D} \) payment rule, bidder A has to pay \( \max(c, b) - bp \) and bidder B has to pay \( \max(c, a) - a \). \( \square \)
Bibliography


