Essays on Corporate Finance Using an Auction Approach

by

Tingjun Liu

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Dissertation Committee:

Professor Christine Parlour (chair)
Professor Richard Green
Professor Burton Hollifield
Professor Fallaw Sowell

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Abstract

My dissertation consists of three essays on corporate finance with a common auction component. The first essay studies how takeover bidding reveals bidders’ private information. The second essay studies the interactions between hedging and bidding when firms compete for the right to a risky project with a specific timing friction. The third essay studies the dual problem of a standard auction.

Chapter one studies takeover bidding when bidders with private synergy values have incentives to signal high values to the market. Such an incentive could arise if the winner sells part of the joint company or issues equity to finance the purchase after the takeover and the stock price includes his value. I show there are two ways a bidder signals a high value. One way is by offering a premium in the opening bid, which results in separating equilibria. The opening bid premium strictly increases in the bidder’s value and is fully revealing. A maximum amount of information is released to the market, leading to efficient allocation and less volatility on the winner’s post-takeover stock price. Alternatively, pooling equilibria exist in which the opening bid is a constant and low, and the winner signals a high value by paying a high final price. The final price does not reveal the exact value, and a minimum amount of information is released to the market. Therefore, the allocation is generically inefficient, leading to larger future resale. In addition, the winner’s post-takeover stock price volatility is larger. Furthermore, a high price signals a high value in the pooling equilibria. Therefore, bidders stay in the auction beyond their actual value, leading to a positive probability of overpayment. Empirically, these two types of equilibria can be distinguished by the size of the initial bid premium. The model predicts novel correlations between the initial bid premium and the winner’s post-takeover stock price performance, volatility, and future resale activity. In both types of equilibria signaling incentives reduce the profits of bidders and increase the profit of the target.

Chapter two is a joint work with Christine Parlour. We investigate a setting in which firms with financing constraints compete in an auction for an indivisible project generating a random cash flow. In addition, firms may invest in an on-going investment opportunity. All firms face a convex cost of external capital and therefore wish to reduce variability in their internal capital. They therefore have an incentive to hedge against the uncertain cash flow generated by winning the auction. However, firms cannot hedge conditional on the outcome. Therefore, hedging leads to business risk: Firms that lose in the auction are exposed to a common factor. Thus, conditional on hedging, firms are worse off if they lose the auction. We characterize optimal hedging and bidding in the case of quadratic investment opportunities and demonstrate how access to financial markets exacerbates financing frictions. In particular, we show that hedging increases the variance of bids. In addition, hedging increases dispersion in the ex-post firm values. Further with hedging the covariance of internal capital changes with the risk factor is negative, and is more negative, the higher the correlation of the hedging instrument with the risk factor. We also show that business risk reduces the seller’s profit, and this has both normative and positive implications.
Chapter three is a joint work with Christine Parlour. We model an auction in which a seller raises a fixed amount of revenue by selling as little as possible of a divisible good. For example, an entrepreneur sells shares in his company to raise a fixed amount of money from venture capitalists. If buyers’ valuations are private, under a simple transformation, there is an equivalence with the standard auction problem. Consequently, the bidding strategies are similar, the revenue equivalence principle translates into the quantity equivalence principle, and the concept of virtual valuation translates into that of virtual quantity in the optimal mechanism design. However, if valuations are interdependent, such equivalence breaks down: In particular, symmetric and increasing pure strategy equilibria do not always exist in a first-price auction, and the linkage principle no longer holds.
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Chapter 1

Takeover Bidding With Signaling Incentives

1.1 Introduction

On November 15, 2006, US Airways’ stock price rose 17% after the airline made a bid of $8 billion for Delta, at a reported 40% premium over Delta’s average trading value over the past thirty days. This paper investigates why a bidder sometimes offers a high initial premium in a takeover, how the market reacts to such an offer, and what consequences the initial premium has on the allocative efficiency and the winner’s post-takeover stock price performance and volatility.

Takeover bidding contests are similar to auctions in which bidders place successively higher bids until the last bid stands. Empirically, a robust feature is the high premia bidders offer (Bradley [1980] and Bradley, Desai and Kim [1988] among others). The high premia in themselves would not be surprising if they resulted from competition that bids up the price. However, what is striking in many takeovers is that bidders offer high premia on the initial bids rather than making a low bid and being prepared to raise it if there is competition. This phenomenon is referred to as jump bidding in the auction literature and is commonly observed in takeovers.

In recent empirical work Betton and Eckbo (2000) find that the initial (and final) offer premium, over the target’s market value sixty days prior to the offer, averages 51% in single-bidder contests, whereas in multi-bidder contests the offer premium averages 45% in the initial

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1 The offer was $4 billion in cash and 78.5 million shares of US Airways stock, valued at $4 billion as of the close the day before the offer, for all shares in a future Delta that would emerge from bankruptcy proceedings. This offer was a 40% premium over the average trading price for Delta unsecured claims over the past thirty days (Bloomberg, Nov 15, 2006; WSJ, Nov 16, 2006).
bid, 65% in the second bid, and 74% in the final bid. These data contain a number of important features. The initial bid premia in both single- and multi-bidder contests are large and the difference between them is relatively small. The second bid in multi-bidder contests is at a relatively large premium over the first bid, and the final price in multi-bidder contests is higher than in single-bidder contests.

The high initial premium and its variation in single- and multi-bidder contests is usually attributed to preemptive bidding (Fishman [1988]) in which a high initial bid preempts a potential rival and results in a single-bidder contest. This theory of preemption explains the large initial premium in single-bidder contests; however, it also predicts the initial premium in multi-bidder contests should be significantly lower than in single-bidder contests, contrary to the empirical findings. In addition, the preemptive theory predicts the final price in a single-bidder contest exceeds that in a multi-bidder contest because the first bidder has a low value in a multi-bidder contest and hence will exit early. However, the data do not support this prediction.

In addition to the preemptive theory, Daniel and Hirshleifer (1997) examine a model of jump bidding with bidding costs, and Avery (1998) investigates a model of jump bidding in the context of affiliated values. Both of these models are developed in the framework of two bidders and do not explain the entire data. The next section on related literature will discuss these models further.

I present a parsimonious model with closed-form solutions that provides a possible explanation for the stylized facts: (1) the high initial premia in both single- and multi-bidder contests, (2) the small difference between them, (3) the relatively large second-bid premium in multi-bidder contests, and (4) the higher final price in multi-bidder contests. In addition, I explore implications of the initial bid premium on the market reaction, the allocative efficiency, and the winner’s post-takeover stock price performance and volatility.

In my model bidders with private synergy values for the target participate in an ascending price auction. Here the synergy is interpreted broadly as the value enhancement created either in the joint company of the bidder and the target or in the target alone through efficiency improvement. All bidders are informed of their synergies and place bids sequentially. No bidding costs exist, but bidders have incentives to signal high synergies to the market through the bidding. Such an incentive arises in any situation where the bidder benefits if the market perceives a high synergy, for example, if the winner sells part of the joint company or issues equity to finance the purchase after the takeover and the stock price includes the synergy. Signaling incentives may also arise from CEO incentives, bidders’ liquidity shocks, and the use of contingent equity.

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2A small portion of the data in Betton and Eckbo (2000) includes situations where the first bidder raises his own bid after the initial bid. These data are excluded from the bids.
payments. Section 1.3 gives details on how signaling incentives arise and how they are modeled.

I note in passing that signaling through the bidding is sensible for the bidders. First, it is credible because it is costly, as will be discussed shortly. Second, it reveals the magnitude of a bidder’s synergy but not its sources, which protects the bidder from possible competitions. Third, it is particularly appropriate if bidders wish to reveal high synergies quickly, for example, if they plan to issue equity upon winning.

The model predicts both separating and pooling equilibria in which a bidder communicates a high synergy to the market through the bidding. In the separating equilibria, a bidder signals a high synergy by offering a premium in his opening bid over the last bid. The bid premium strictly increases in the bidder’s synergy and is fully revealing. The bid premium is a credible signal of the bidder’s synergy because overbidding is costly as the bidder may win with it. In fact, I show that even small signaling incentives may result in a significant initial premium. The separating equilibria correspond to the maximum amount of information release because the opening bid fully reveals the private information. Consequently, bidders exit at their actual synergies, the allocation is efficient, and the winner’s post takeover stock price volatility is smaller.

The separating equilibria capture the observed bidding characteristics. In a parametric example the model predictions match the data well. Bidders’ incentives to signal high synergies to the market explain the observed high initial premia in single- and multi-bidder contests and the relatively large second-bid premium in multi-bidder contests. In addition, the model predicts the observed feature that the initial premium in a single-bidder contest is slightly higher than in a multi-bidder contest because of an ex-post selection bias that a high initial premium is more likely to result in a single-bidder contest as it may exceed the synergies of all other bidders. Furthermore, the final price in multi-bidder contests is higher than in single-bidder contests, which is consistent with the data and is contrary to the preemptive theory because the first bidder’s synergy conditional on being a multi-bidder contest is not necessarily low and thus will remain in the auction until a higher price obtains.

The pooling equilibria have completely different properties. The opening bid is low and independent of the synergy, and thus it conveys no information. Instead, the market infers the winner’s synergy through the final price. However, the final price only partially reveals the winner’s synergy because the highest losing bidder determines the price in an ascending price auction. In other words, the only information the market has on the winner is that he is willing to pay at least the final price. Consequently, the pooling equilibria correspond to the smallest amount of information release. Thus the allocation is generically inefficient, leading to larger

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More discussions on how the initial premia differ in single- and multi-bidder contests follow in the main analysis.
future resale. The winner’s post-takeover stock price volatility is greater.

A high price signals a high synergy in the pooling equilibria. Therefore, bidders stay in the auction past their actual synergies, leading to possible overpayment. A positive probability exists for a decrease in the winner’s stock price after the takeover.

These two types of equilibria have very different properties in multiple dimensions. Furthermore, they can be distinguished empirically because the initial bid premium in any pooling equilibrium is smaller than in any separating equilibrium. Therefore, the model predicts observable correlations between the initial bid premium and the winner’s post-takeover stock price performance, volatility, and future resale, generating a rich set of novel empirical predictions. Some of the predictions may be counter-intuitive; for example, casual intuition may suggest a high initial bid is more likely to result in an overpayment. This model predicts, however, that precisely because bidders have signaled their synergies through a high initial bid, they have no need to signal again with an overpayment. Therefore, conditional on observing a larger initial premium, observing a decrease in the winner’s post-takeover stock price is less likely.

In both types of equilibria, signaling incentives reduce the profits of bidders and increase the profit of the target. In the separating equilibria bidders open high but exit truthfully, whereas in the pooling equilibria bidders open low but stay past their synergies. The intuition for the reduced bidder profit (and hence increased target profit) is that bidders receive rents for their private synergies but not on the portion that is signaled and made public.

Since signaling incentives increase the profit of the target and decrease the profit of a bidder, this model suggests the target (or a bidder) can benefit by controlling the signaling incentives. The target should request contingent equity payments as opposed to cash payments. For the bidder using internal capital or issuing debt is best, as opposed to issuing equity, in order to finance the purchase. The preferred means of financing for the bidder is the same as in the pecking-order theory (Myers and Majluf [1984]), but the reasons are different. In the pecking-order theory, the choice of financing is endogenous and equity issuance is itself a signal that the bidder is overvalued. On the other hand, the choice of financing in my model is exogenous, and equity issuance reduces the bidder’s profit because it forces the bidder to signal his private value in the preceding auction. Nonetheless, the same preference ordering over financing obtains.

The rest of the paper is organized as follows. Section 1.2 contains a more detailed discussion of related literature, Section 1.3 describes the model, Section 1.4 analyzes the equilibria, Section 1.5 discusses the target’s and bidders’ profits, and Section 1.6 first uses a parametric example to match the observed bidding characteristics and then derives general empirical implications.

Examples of contingent payments are given in Section 3 when signaling incentives are discussed in more detail.
Finally, I present a conclusion in Section 1.7. All proofs are in the first appendix, Section 1.8. Some comments on the equilibria under relaxed assumptions follow in the second appendix, Section 1.9. A table listing the frequently used notation is given in the third appendix, Section 1.10.

1.2 Related Literature

The jump bidding model of Fishman (1988) has one bidder and a potential rival who pays an investigation cost to learn his private value for the target. If the bidder has a high value, he signals this value by placing a high initial bid that preempts the rival and results in a single-bidder contest; if his signal is low, he places the minimum required bid that does not deter the rival, leading to a multi-bidder contest. His model explains the observed high initial premium in single-bidder contests but does not explain other aspects of the data.

In the jump bidding models of Daniel and Hirshleifer (1997) and Avery (1998), an initial bid premium signals a high value and the auction ends immediately if rival bidders perceive that their values are lower than the signaled value. Avery (1998) has no bidding costs but uses weakly dominated strategies because bidders exit early. Both of these models are developed in the framework of two bidders, and how the results vary when the number of bidders changes is unclear. By contrast, my model applies to an arbitrary number of bidders and can explain the bidding features in both single- and multi-bidder contests. In addition, my model assumes no bidding costs and no weakly dominated strategies played in equilibrium, which means bidders stay in the competition as long as surplus remains.

My work is also related to existing literature that gives rise to a positive probability of overpayment with rational agents who observe their precise synergy values. The overpayment arises from the strategic role of debt (Chowdhry and Nanda [1993]) or the bidder toeholds (Burkart [1995], Singh [1998], Bulow, Huang, and Klemperer [1999]). My work identifies another framework with positive probability of overpayment in which bidders are willing to pay a high price to signal a high synergy in the pooling equilibria. Furthermore, my model predicts the probability of overpayment is negatively correlated with the initial bid premium.

Empirically, circumstantial evidence of possible overpayment exists. A large literature studies bidders’ post-takeover performance, as documented in, for example, Asquith (1983), Schwert (1996), Loughran and Vijh (1997), and Mitchell and Stafford (2000). The findings are mixed but suggest the possibility of a long-term downward drift in the winner’s market value after the takeover. One interpretation is that bidders overpay and that it takes the market some time to
gradually learn about this mistake (Schwert [2003]). In addition, event studies on takeovers have also found that the market considers many acquisitions “bad news” for the bidders (Berkovitch and Narayanan [1993], for example).

As discussed in Section 1.3, signaling incentives are modeled by assuming the winner sells a fraction of the target after the takeover. This feature makes my model resemble an auction with a resale market. Bikhchandani and Huang (1989) examine discriminatory and uniform-price auctions in treasury bill markets with pure common value and a secondary market. The auction winner sells the object in the secondary market and the price depends on information revealed in the primary auction. Thus, bidders have incentives to signal high values. They find this incentive increases the bids and reduces bidders’ profits. I find similar results in that bidders’ profits are reduced because they either open high or are willing to pay a high price, although I focus on pure private values instead. My model differs from theirs in that each bidder arrive sequentially and places multiple bids in an ascending price auction. This approach is more appropriate for takeovers. Furthermore, it allows the study of the bid dynamics and gives rise to multiple signaling equilibria with different implications on the initial bid premium, allocative efficiency, and post-takeover stock price performance and volatility.

1.3 The Model

A group of $n$ risk-neutral bidders bids with cash for a target. The interest rate is zero and there are four dates, $t = 0, 1, 2, 3$. At $t = 0$, bidder $i$ learns $s_i$, his private synergy by which the target’s market value can be enhanced, independently drawn from distribution $F$ on $[s, \overline{s}]$ with $s \geq 0$. He also learns whether he is active, with an independent probability $q_i$, or not, where $0 < q_i \leq 1$. Only active bidders can participate in the bidding. The first bidder is always active such that $q_1 = 1$.

The target’s and all bidders’ market values before the takeover are public, and the only private information is bidders’ synergies. Bids are in increments over the target’s market value before the takeover. The target accepts the best offer above a reserve price $\underline{s}$. The auction starts at $t = 1$ in two stages. In the first stage bidders sequentially arrive at the market in a predetermined order and each bidder places an opening bid. Let $b_i$ denote bidder $i$’s opening bid

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5I have abstracted away from the free-rider problem (Grossman and Hart [1980]) by assuming, for example, that bidders can sufficiently dilute the target’s assets after the takeover. The reserve price is assumed to be $\underline{s}$ for simplicity; different reserve values do not change the qualitative results.

6I will show the expected profit of a bidder is independent of the order. Thus, the equilibrium is consistent with an endogenous order.
Bidder learns \( s_i \)

receives permission
to participate

Bidding in

two stages

Winner sells \( \mu_i \)

\( s_i \) partially revealed

Winner sells \((1-\mu_i)\)

\( s_i \) fully revealed

\[ \downarrow \]

\[ t = 0 \]

\[ \downarrow \]

\[ t = 1 \]

\[ \downarrow \]

\[ t = 2 \]

\[ \downarrow \]

\[ t = 3 \]

Figure 1.1: Sequence of events

and let \( b_{i-1} \equiv \{b_1, \ldots, b_{i-1}\} \) denote the sequence of previous opening bids. If bidder \( i \) is inactive or does not wish to participate in the bidding, he submits a nonparticipating bid of \(-1\) and is excluded from the rest of the auction; otherwise, he submits a bid greater than or equal to the previous highest bid \( \max \{b_{i-1}\} \). For the first bidder the previous highest bid is defined to be \( \max \{b_0\} \equiv s \).

To facilitate later discussions we define an opening premium for each participating bidder. For the first bidder it is the difference between his opening bid \( b_1 \) and the target’s market value before the takeover. The first bidder’s opening premium is also called the initial premium. The opening premium for other participating bidder \( i > 1 \) is the difference between his opening bid \( b_i \) and the previous highest bid \( \max \{b_{i-1}\} \).

If there is only one participating bidder at the end of the first stage, he wins with his opening bid and the result is a single-bidder contest. On the other hand, a multi-bidder contest follows if two or more bidders participated in the first stage. They then proceed to compete in the second stage in the form of a standard ascending price auction in which price continuously rises from the highest bid in the first stage and bidders exit until only the winner remains.

At \( t = 2 \), if bidder \( i \) is the winner, he sells an exogenous fraction \( \mu_i \in (0, 1) \) of the target to the market. His synergy \( s_i \) is partially revealed at this time. Specifically, with an exogenous probability \((1 - r_i)\), where \( 0 < r_i \leq 1 \), \( s_i \) is revealed to the market through means independent of the bidding process, whereas with probability \( r_i \), the market has to infer \( s_i \) through the observed bids. In the long run, at \( t = 3 \), \( s_i \) is fully revealed to the market, and the winner sells the remaining fraction \((1 - \mu_i)\). Figure 1.1 depicts the sequence of events.

The assumption of a random number of active bidders is for added flexibility in matching the model predictions with empirical observations. An example of an inactive bidder is one with financial constraints or alternative investment opportunities. Econometricians do not observe the nonparticipating bids, but I assume bidders do. The latter assumption is not important for the
quantitative results of the paper, and it only simplifies bidders’ inference problem in the opening stage on the number of remaining active bidders. The assumption that price continuously rises in the second stage is made to simplify the analysis and is equivalent to assuming bidders outbid each other only in small increments after the opening stage. This assumption is qualitatively consistent with observations in Betton and Eckbo (2000) that bid increments after the opening stage are significantly less than the initial premium.

The signaling incentives are modeled by assuming the synergy is in the target alone and the winner sells a fraction of the target after the takeover. This assumption is equivalent to assuming the winner sells a fraction of the joint company that includes the synergy. This assumption captures many types of signaling incentives either in an exact or a reduced form.

First, if the winner can increase the target’s value but has no interest in running it himself, he may sell part or all of the target shortly after the takeover. Second, the bidder’s CEO may act in his own interest, and his compensation may depend on the bidder’s short-term performance after the takeover (for example, if he is paid with options), which reflects the synergy the market perceives. If the compensation is linear in both the short- and long-term performance, the relative weight of the short-term component corresponds to the sell-off fraction used in the model. A third possibility is that the bidder, or part of the bidder’s shareholders, may face liquidity shocks and need to sell the shares shortly after the takeover (see, for example, Miller and Rock [1985] and Teoh and Hwang [1991]). If the management acts in the best interest of the stockholders, it will try to signal a high synergy to the market. In this case the sell-off fraction is the probability of the bidder, or the fraction of the stockholders, facing liquidity shocks.

The model captures the signaling incentives in the above situations in an exact form. In addition, it is a reduced form approximation for the signaling incentives in many other situations. One such situation is where bidders issue equity to fund part or all of the purchase upon winning, and a high post-takeover stock price is desirable. One variation of this is the so-called cash equivalent of equity payment, where the bidder makes an acquisition with stock and a floating exchange ratio that guarantees the cash value of the offer. The cash equivalent of equity payment is the same as bidding with cash and a commitment of issuing equity to finance the entire purchase. It is an extreme case of the so-called collar clause contained in many takeovers with equity payments, which stipulates a range such that if the dollar value of the offer, calculated at the time of deal closing, goes outside the range, an adjustment in the payment will be made.7 Furthermore, when bidders bid with equity, even without collar clauses, they still have incentives

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7Discussions on the cash equivalent of equity payment (although this name is not used) and collar clause can be found in Gaughan (2002).
to signal high values because their offers will be more attractive if their stock prices are high.\textsuperscript{8} In all of the above situations, bidders profit if the market perceives a high synergy, which is the source of the signaling incentives.\textsuperscript{9} Empirically, these situations apply to a significant fraction of takeovers. Andrade, Mitchell and Stafford [2001] document an increasing and overwhelming use of stock payments in takeovers and they report that about 70\% of all deals in the 1990s involved stock compensation with 58\% entirely stock financed, representing approximately 50\% increase over the numbers in the 1980s.

\subsection{1.3.1 The Market’s Belief and Bidding Strategies}

At $t = 2$, if the winner’s synergy is not independently revealed, the market forms belief about it based on the observed bidding history. Let the set $h_t$ be the bidding history at time $t$, which includes all the opening bids, exit prices, and their associated bidder identities up to that time.\textsuperscript{10} Let $\pi_i$ denote any bidder $i$’s expected profit. If $i$ wins, let $\theta_i(h_2)$ denote the market’s belief about $s_i$ at $t = 2$. Then the winning profit is the sum of the profits from $t = 2$ and $t = 3$:

\[
\pi_i = \mu_i (1 - r_i) (s_i - p) + \mu_i r_i (\theta_i(h_2) - p) + (1 - \mu_i) (s_i - p)
\]

\[
= \mu_i r_i \theta_i(h_2) + (1 - \mu_i) s_i - p,
\]

where $p$ is the final price. The winner’s profit is a weighted sum of his synergy and the market’s belief about it. The dependence on the market’s belief is the source of signaling incentives, and I define the corresponding coefficient, denoted by $\lambda_i \equiv \mu_i r_i$, to be the \textit{signaling incentive} for bidder $i$. Intuitively, the signaling incentive is the product of the sell-off fraction and the probability that the synergy is not revealed.

\textsuperscript{8}Auctions with equity payments can be different depending on how the winner is determined. Hansen (1985) studies an auction with equity payments in which the winner is the one who bids the largest fraction. This paper only models cash auctions explicitly, but it applies to auctions with equity payments in which the winner is the one whose offer has the largest cash value. This winner selection rule is relevant to takeovers where a capital market exists, and this makes an auction with equity payments similar to a cash auction in which the winner issues equity to finance the bid after winning.

\textsuperscript{9}Although the model does not exactly describe the above situations, the author has explicitly examined equilibria in the case of cash equivalent of equity payment (not included in this paper) and finds that the qualitative results of the paper go through.

\textsuperscript{10}One can conveniently assume the bidding lasts for a finite time from $t = 1$ to $t = 1 + \epsilon_1$ in the first stage and from $t = 1 + \epsilon_1$ to $t = 1 + \epsilon_2$ in the second stage, where $0 < \epsilon_1 < \epsilon_2 < 1$ such that information is released gradually over time.

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Then $i$’s winning profit becomes

$$
\pi_i = \lambda_i \theta_i (h_2) + (1 - \lambda_i) s_i - p.
$$

If $i$ loses, $\pi_i = 0$.

Bidders maximize their expected profits through optimal bidding strategies. Recall that the strategy of an inactive bidder is trivial: open with a nonparticipating bid. For brevity I will implicitly assume the bidder to be active when discussing his strategies. The opening bid in general depends on the bidder’s synergy $s_i$ and the bidding history, which consists of the previous opening bids $b_i^{j-1}$. Let $\beta_i (s_i|b_i^{j-1})$ denote the opening strategy. The bidder will not participate if $s_i$ is too low, and a participation threshold exists, as a function of $b_i^{j-1}$, such that the opening bid is nonparticipating if $s_i$ is less than the threshold, and the opening bid is greater or equal to the previous highest bid $\max \{ b_i^{j-1} \}$ if $s_i$ is above the threshold.

In the second stage a bidder determines a price at which to exit if the auction is still ongoing, at every point of time, as a function of his synergy and the bidding history. Let $\gamma_i (s_i|h_t)$ denote the exit price, and this is the exit strategy for $i$. By definition the exit price is at least the opening bid, or $\gamma_i (s_i|h_t) \geq \beta_i (s_i|b_i^{j-1})$, for all $s_i$ and $h_t$. The doublet $\{ \beta_i (s_i|b_i^{j-1}), \gamma_i (s_i|h_t) \}$ constitutes the overall strategy for $i$.

The market’s belief must be within the lower and upper synergy bounds $\underline{s}$ and $\bar{s}$. Additionally, I impose a rational expectations condition that the bidding strategy and the market’s belief be consistent on the equilibrium path. That is, the market’s belief about the winner’s synergy is its expected value conditional on all the information the market has on the winner, which includes the winner’s (observed) opening bid and (unobserved) exit price being greater than or equal to the final price\(^{11}\). Or

$$
\theta_i = \mathbb{E} [ s | \beta_i (s|b_i^{j-1}) = b_i, \gamma_i (s|h_t) \geq p \text{ for all } t \in [1 + \epsilon_1, 1 + \epsilon_2] ],
$$

where the expectation is taken over the signal distribution $F$, and $[1 + \epsilon_1, 1 + \epsilon_2]$ is the duration of the second stage bidding as explained in a previous footnote.

Equation 1.1 contains the main intuition for this paper. The market’s belief generally depends on both the bidder’s opening bid and the final price. I will show the dependence on the opening bid gives rise to opening premia and the dependence on the final price gives rise to possible overpayment. Furthermore, these two situations correspond to two types of equilibria, and they

\(^{11}\)As a subtle point, no second stage of bidding exists in a single-bidder contest. One can think of the winner’s exit price as being infinitely high in this situation.
are different because the opening bid reveals the bidder’s exact opening strategy, but the final price gives the market only a noisy estimate of the winner’s exit strategy because all the market knows is that the winner is willing to pay at least the final price. In other words, the exit strategy of the highest losing bidder is fully revealed, but the winner’s is not. I will show that this difference causes the two types of equilibria to have completely different properties across multiple dimensions, generating a rich set of empirical predictions.

The market’s belief affects the winner’s post-takeover stock price movements. To proceed, let \( v_0 \) denote the winner’s market value before the takeover. Let \( v_2 \) and \( v_3 \) denote his short- and long-term post-takeover market values at \( t = 2 \) and \( t = 3 \) respectively. The value for \( v_2 \) is simply \( v_0 \) plus the difference between the actual synergy, or the market’s belief about it, and the price:

\[
v_2 = \begin{cases} 
  v_0 + s_i - p & \text{if } s_i \text{ revealed at } t = 2 \\
  v_0 + \theta_i - p & \text{otherwise},
\end{cases}
\]  

(1.2)

where \( i \) is the winning bidder. The value for \( v_3 \) is \( v_0 \) plus the sum of proceeds from selling at \( t = 2 \) and \( t = 3 \), less the price:\[
v_3 = \begin{cases} 
  v_0 + s_i - p & \text{if } s_i \text{ revealed at } t = 2 \\
  v_0 + \mu_i \theta_i + (1 - \mu_i) s_i - p & \text{otherwise}.
\end{cases}
\]  

(1.3)

1.3.2 Equilibria Refinement and Characterization

Signaling models in general have multiple equilibria due to freedom in specifying off the equilibrium path beliefs. The situation is particularly complicated in this model because bidders arrive sequentially, and thus both their bidding strategies and the market’s belief are history dependent. In order to focus on the effects of signaling incentives, I will make a number of assumptions to refine the set of equilibria. The assumptions are imposed on the equilibrium strategy and the market’s belief; therefore, they serve to select a subset of equilibria of interest. However, the restricted equilibria are still equilibria within the unrestricted class.

In the most general form, the market’s belief about the winner’s synergy depends on all opening bids and exit prices. However, since synergies are private and independent, it is reasonable to assume that the market’s belief about any bidder’s synergy should not in principle depend on the actions of other bidders. More precisely, recall that the winner \( i \)’s opening bid \( b_i \) is a function of his synergy \( s_i \) and the previous opening bids \( b_i^{-1} \); therefore, the market can infer \( s_i \)

\[12\] One exception is when signaling incentives come from the CEO acting in his own interest. In such a case the bidder does not sell a fraction at \( t = 2 \) and \( v_3 = v_0 + s_i - p \) regardless of whether the synergy is revealed at \( t = 2 \). The corresponding analysis for this situation does not change the general features of the results.
through $b^{i-1}$ and $b_i$. Furthermore, the final price $p$ is also informative of $s_i$ because the winner’s exit price, which is a function of $s_i$, is at least $p$. I now make a simplifying assumption that the market’s belief about the winner $i$’s synergy depends only on $b_i$, $b^{i-1}_i$, and $p$.

**Assumption 1.** The market’s belief about the winner $i$’s synergy $s_i$ only depends on his opening bid $b_i$, the previous opening bid $b^{i-1}_i$, and the final price $p$.

This assumption rules out dependence of the market’s belief about the winner’s synergy on any opening bids after him or on any exit prices except the exit price of the last losing bidder, which is the final price. This assumption is necessary because the market’s belief off the equilibrium path is otherwise unrestricted.

In light of assumption 1, let $\theta_i (b_i, p | b^{i-1})$ denote the market’s belief about the winner $i$’s synergy. For notational compactness in the rest of the paper, I suppress the arguments and use $\theta_i$ wherever doing so causes no confusion.

I simplify the bidder’s exit strategy under assumption 1. In the second bidding stage, a bidder $i$ maximizes his expected profit, which depends on his synergy, the market’s belief about his synergy conditional on his winning, and his exit price. Because the market’s belief depends on $b_i$, $b^{i-1}_i$, and $p$, and because $b_i$ is a function of $s_i$ and $b^{i-1}_i$, the exit strategy can only depend on $s_i$ and $b^{i-1}_i$. Therefore, I use $\gamma_i (s_i | b^{i-1})$ to denote bidder $i$’s exit strategy.

The consistency requirement between the bidding strategy and the market’s belief in equation 1.1 simplifies to

$$\theta_i = E[s | \beta_i (s | b^{i-1}) = b_i, \gamma_i (s | b^{i-1} \geq p)].$$

As a further restriction on the exit strategy, I assume, as is standard in auction literature, no bidder uses weakly dominated strategies in equilibrium. This assumption is necessary to make the equilibrium robust against trembling hands.

**Assumption 2.** No weakly dominated strategies are played in equilibrium.

Assumption 2 rules out bidders exiting when surplus remains. This assumption, together with the assumption of no bidding costs, sets my model apart from Fishman (1988) and Avery (1998).

I also impose another standard assumption that the equilibrium opening and exit strategies are nondecreasing.

**Assumption 3.** The equilibrium opening and exit strategies are nondecreasing; that is, for all bidder $i$ and previous opening bids $b^{i-1}$, the opening strategy $\beta_i (s_i | b^{i-1})$ and the exit strategy $\gamma_i (s_i | b^{i-1})$ are nondecreasing in the synergy $s_i$. 

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Since the market’s belief satisfies equation 1.4 on the equilibrium path, under assumption 3 it is straightforward to show that the market’s belief is nondecreasing in the final price on the equilibrium path (the proof is omitted). I naturally impose this requirement on all paths.

**Assumption 4.** The market’s belief about the winner’s synergy is nondecreasing in the final price. That is, for any opening bid \( b_i \) of bidder \( i \), any previous opening bids \( b_i^{-1} \), and any two prices \( p'' \geq p' \), the market’s belief \( \theta_i(\{b_i, p''|b_i^{-1}\}) \geq \theta_i(\{b_i, p'|b_i^{-1}\}) \).

I now characterize the exit strategy. In a standard ascending price auction with private values, bidders exit at their truthful values. With signaling incentives the exit strategy becomes a weighted sum of the synergy and the market’s belief about it, and intuitively the weight on the market’s belief is the signaling incentive.

**Proposition 1.** In any equilibrium satisfying assumptions 1 through 4, the exit strategy for a participating bidder \( i \) with synergy \( s_i \) is

\[
\gamma_i(s|b_i^{-1}) = (1 - \lambda_i)s_i + \lambda_i E[s|\beta_i(s|b_i^{-1}) = \beta_i(s|b_i^{-1}), s \geq s_i].
\] (1.5)

Heuristically, the exit strategy in equation 1.5 is determined by the basic intuition that the bidder is indifferent between winning and losing at the exit price. If bidder \( i \) wins at the exit price, his net profit must be zero. The total proceeds from selling at \( t = 1 \) and \( t = 2 \) are

\[
(1 - \lambda_i)s_i + \lambda_i E[s|\beta_i(s|b_i^{-1}) = \beta_i(s|b_i^{-1}), \gamma_i(s|b_i^{-1}) \geq \gamma_i(s|b_i^{-1})]\.
\]

The proof of proposition 1 shows that \( \gamma_i(s|b_i^{-1}) \) is strictly increasing. Therefore, the above simplifies to

\[
(1 - \lambda_i)s_i + \lambda_i E[s|\beta_i(s|b_i^{-1}) = \beta_i(s|b_i^{-1}), s \geq s_i].
\]

The proceeds should equal the payment \( \gamma_i(s|b_i^{-1}) \), yielding equation 1.5.

Proposition 1 offers intuition for understanding the two modes of signaling. The exit strategy depends on whether equation \( \beta_i(s|b_i^{-1}) = \beta_i(s_i|b_i^{-1}) \), where \( s \) is the unknown, has a unique solution. First note that the equation has at least one solution at \( s = s_i \). If this is the unique solution, the opening bid reveals the bidder’s exact synergy and I say the opening strategy is *separating* at \( s_i \). In this case the exit strategy in equation 1.5 reduces to \( \gamma_i(s_i|b_i^{-1}) = s_i \). Bidders exit truthfully because they have already signaled to the market their precise synergies with the opening bids.

On the other hand, if the solution is not unique, the opening bid does not reveal the bidder’s exact synergy and I say the opening strategy is *pooling* at \( s_i \). Since \( \beta_i(s|b_i^{-1}) \) is nondecreasing,
the solution to $\beta_i (s | b_i^{j-1}) = \beta_i (s_i | b_i^{j-1})$ must be an interval that includes $s_i$. Assume the solution is $[s_1, s_2]$, where $s_1 \leq s_i \leq s_2$, then equation 1.5 becomes $\gamma_i (s_i) = (1 - \lambda_i) s_i + \lambda_i E[s | s_i \leq s \leq s_2]$. In this case $\gamma_i (s_i) \geq s_i$, and the inequality is strict except at the upper limit $s_i = s_2$. Therefore, in a pooling equilibrium a bidder optimally stays past his actual synergy because the opening bid does not fully reveal the synergy and the bidder uses a higher price to indicate a higher synergy.

The above intuition is summarized below.

**Corollary 1.** In any equilibrium satisfying assumptions 1 through 4, the exit price is at least the bidder’s synergy, or

$$\gamma_i (s_i | b_i^{j-1}) \geq s_i$$

for all $i$. Equation 1.6 holds as an equality if and only if $s_i = \max \{ s : \beta_i (s | b_i^{j-1}) = \beta_i (s_i | b_i^{j-1}) \}$.

Corollary 1 shows the exit price is either greater than or equal to the actual synergy depending on whether the opening strategy is separating or pooling. This dependence has implications on the allocative efficiency. To proceed I first define efficiency in the model.

**Definition 1.** An equilibrium is efficient if and only if the winner is always the one with the highest synergy among all active bidders.

Efficient allocation maximizes social welfare, and inefficient allocation induces observable future resale. A number of factors affect efficiency in this model. First of all, if the opening strategy is pooling on part or all of $[s, \bar{s}]$, the exit price in equation 1.5 depends on both the synergy and the signaling incentive, and bidders with the same synergy will exit at different prices if their signaling incentives differ. Therefore, the combination of heterogeneity in the signaling incentives and the pooling equilibrium is a source of inefficiency.

Furthermore, even if all bidders have the same signaling incentives, but if their opening strategies have different separating and pooling patterns, that is, if one bidder’s opening strategy is pooling and another’s is separating at some synergy, their exit strategies will differ, creating inefficiency. However, this type of inefficiency can be removed if the opening strategies of all bidders are consistent, which I impose below.

**Assumption 5.** The opening strategies of all bidders are consistent. If bidder 1’s opening strategy is separating or pooling on any interval, any other participating bidder’s opening strategy is also separating or pooling on that interval. Or formally, for all bidder $i > 1$, all previous opening bids $b_i^{j-1}$, and any two synergies $s' < s''$, (i) if $\beta_1 (s') = \beta_1 (s'')$, then $\beta_i (s' | b_i^{j-1}) = \beta_i (s'' | b_i^{j-1})$; and (ii) if $\beta_1 (s') < \beta_1 (s'')$, then either $\beta_i (s' | b_i^{j-1}) < \beta_i (s'' | b_i^{j-1})$ or $\beta_i (s' | b_i^{j-1}) = \beta_i (s'' | b_i^{j-1}) = -1$. 

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To understand this assumption note that in a simultaneous auction where bidders submit opening bids at the same time, it is natural to assume that two bidders with the same synergy make the same bid. However, when bidders arrive sequentially, two bidders with the same synergy may face different history and may bid differently. Therefore, assumption 5 can be interpreted as a relaxed version of the requirement that bidders with the same synergy make the same bid.

As can be seen from proposition 1, assumption 5 ensures that bidders with the same signaling incentive will employ the same exit strategy provided they participate. However, inefficiency may still arise if the market’s belief off the equilibrium path is overly pessimistic so as to discourage a bidder with high synergies from participating. Consider the following pathological example for illustrative purposes:

Example 1. Suppose the synergy lower and upper bounds are 0 and 1 respectively, bidder 1’s opening bid \( b_1 = 0.5 \) and bidder 2’s signaling incentive \( \lambda_2 = 0.75 \). If the market’s belief about bidder 2’s synergy, if he wins the auction, is such that \( \theta_2(b_2, p|b_1 = 0.5) = 0 \) for all \( b_2 \) and \( p \), then bidder 2 will never participate regardless of his synergy.

To verify that for bidder 2 not to participate is optimal, note that if bidder 2 participates and wins, the final price will be \( p \geq b_1 = 0.5 \) and his expected profit will be \( \pi_2 = (1 - \lambda_2) s_2 - p \leq 0.25 - 0.5 < 0 \), thus confirming the claim.

One can verify that this example satisfies all of the above assumptions. For example, the consistency requirement in equation 1.4 on the market’s belief is trivially satisfied because the requirement is restrictive only on the equilibrium path but the bidder never participates in equilibrium.

Therefore, it is necessary to impose a lower bound on the market’s belief such that bidders with high synergies are not deterred from participating. To motivate such a restriction, note that a lower bound naturally exists on the equilibrium path. From equation 1.4 I show the market’s belief about the winner’s synergy is at least the previous highest opening bid, or \( \theta_i \geq \max \{ \overline{b}^{i-1} \} \) on the equilibrium path.

Lemma 2. For any opening bid \( b_i \) of winner \( i \), any previous opening bids \( \overline{b}^{i-1} \), and any final price \( p \geq b_i \), if they are on the equilibrium path, i.e., if the solution to \( \beta_i(s|\overline{b}^{i-1}) = b_i \) and \( \gamma_i(s|\overline{b}^{i-1}) \geq p \) is nonempty, the market’s belief is at least the previous highest opening bid, or \( \theta_i \geq \max \{ \overline{b}^{i-1} \} \). Thus, on the equilibrium path the market’s belief about a participating bidder’s synergy is at least the previous highest opening bid. I now impose this restriction on all paths.

\(^{13}\)One can check that the example also satisfies the intuitive criteria (Cho and Kreps [1987]), a commonly used refinement we do not impose because it does not help.
**Assumption 6.** The market’s belief about the winner’s synergy is no less than the previous highest opening bid. That is, $\theta_i \geq \max \{ \hat{b}^{i-1} \}$ for all $\hat{b}^{i-1}$, $b_i \geq \max \{ \hat{b}^{i-1} \}$, and $p \geq b_i$, where $i$ is the winner.

This assumption ensures that a bidder with a high synergy will participate. All these assumptions together ensure that the equilibrium will be efficient if all bidders have the same signaling incentive.

**Lemma 3.** If an equilibrium satisfies assumptions 1 through 6 and the signaling incentive $\lambda_i$ is the same for all bidder $i$, then the equilibrium is efficient.

### 1.4 Analysis of Equilibria

I solve for the equilibria under assumptions 1 through 6 in this section. For comparison, in the limit when the signaling incentive $\lambda_i$ approaches zero for all bidder $i$, the bidding contest reduces to a standard ascending price auction with a unique equilibrium: All bidders will open with the minimum bid $s$ and exit truthfully.

In the general case with positive signaling incentives, a continuum of equilibria arises. Two simplest types exist: pure separating and pure pooling equilibria. They represent two polar extremes and are the building blocks of all other equilibria.

**Definition 2.** A pure separating equilibrium is one that satisfies assumptions 1 through 6, and for all bidder $i$ and the previous opening bids $\hat{b}^{i-1}$, the opening strategy $\beta_i (s_i|\hat{b}^{i-1})$ is strictly increasing in the synergy $s_i$ if $s_i$ exceeds the participation threshold. A pure pooling equilibrium is one that satisfies assumptions 1 through 6, and for all bidder $i$ and the previous opening bids $\hat{b}^{i-1}$, the opening strategy $\beta_i (s_i|\hat{b}^{i-1})$ is independent of the synergy $s_i$ if $s_i$ exceeds the participation threshold.

I find a continuum of pure separating equilibria, and they differ only trivially. In any such equilibrium, a participating bidder’s opening bid strictly increases in the synergy. The opening bid is a credible signal because it is a commitment of minimum payment. The pure separating equilibria give rise to opening premia and correspond to the maximum amount of information released to the market because the opening bid fully reveals the private synergy. Consequently, bidders exit truthfully, the allocation is efficient, and a smaller post-takeover stock price volatility results.

I also find a continuum of pure pooling equilibria and they differ trivially. In any such equilibrium bidders submit low opening bids, at $s$ or slightly above it, and the bid is independent
of the bidder’s synergy. Thus the opening bid contains no information and bidders signal their synergies with the final price. A high price signals a high synergy. Therefore, bidders stay in the auction past their actual synergy, creating a positive probability of overpayment. The price does not reveal the exact synergy, and the smallest amount of information is released to the market. Consequently, the allocation is generically inefficient and larger post-takeover stock price volatility results.

Any other equilibrium is a hybrid where the opening strategy is separating in part of the synergy range and is pooling elsewhere, and bidders signal with a combination of the opening bid and the final price, corresponding to an intermediate amount of information released to the market. Bidding strategies in hybrid equilibria are generally complicated and the equilibria properties are between the pure separating and pure pooling equilibria.

The model does not predict which equilibria will occur, but nonetheless generates testable results. Specifically, since the pure separating and pure pooling equilibria are different in multiple dimensions, and the hybrid equilibria have properties in between, observable correlations among these dimensions exist that Section 1.6 will discuss.

I will only investigate the pure separating and pure pooling equilibria in the main analysis for simplicity since the hybrid equilibria have properties in between. Formally, I show that the hybrid equilibria involve discontinuous opening strategies. I now impose an additional assumption that the first bidder’s equilibrium opening strategy is continuous.

**Assumption 7.** The first bidder’s equilibrium opening strategy $\beta_1(\cdot)$ is continuous on $[\underline{s}, \bar{s}]$.

Assumption 7 rules out the hybrid equilibria. This assumption is removed in the second appendix, where a qualitative discussion on hybrid equilibria is given.

**Lemma 4.** Any equilibrium satisfying assumptions 1 through 7 is either pure separating or pure pooling.

### 1.4.1 The Pure Separating Equilibria

In the pure separating equilibria, the opening bids of participating bidders are fully revealing and their synergies can be deduced. For bidder $i$, let $s_{i}^{\text{max}}$ denote the highest synergy among the previous participating bidders and define $s_{i}^{\text{max}} \equiv \underline{s}$ for the first bidder. Furthermore, define $G_i(y) \equiv \prod_{k=i+1}^{\infty} (1 - p_k + p_k F(y))$, which is the distribution of the largest synergy among the remaining active bidders. I now state the main result of this section.
Proposition 2. The set of pure separating equilibria is nonempty. The opening and exit strategies for bidder $i$ satisfy the following:

$$
(i) : \beta_i \left(s_i | b_i^{-1}\right) = \begin{cases} 
-1 & \text{if } s_i < \max(b_i^{-1}) \\
\in [\max(b_i^{-1}) , s_i] & \text{if } \max(b_i^{-1}) \leq s_i < s_i^{\text{max}} \\
L_i^{-1}(\lambda_i L_i(s_i)) & \text{if } s_i \geq s_i^{\text{max}}
\end{cases}
$$

(1.7)

$$
(ii) : \gamma_i \left(s_i | b_i^{-1}\right) = s_i,
$$

(1.8)

where $L_i^{-1}(\cdot)$ is the inverse function of $L_i(x) = \int_{s_i^{\text{max}}}^{x} G_i(y) \, dy$. The market’s belief is

$$
\theta_i = \begin{cases} 
\beta_i^{-1} \left(b_i | b_i^{-1}\right) & \text{if } \max(b_i^{-1}) \leq b_i \leq \beta_i \left(s | b_i^{-1}\right) \\
\bar{s} & \text{if } b_i > \beta_i \left(s | b_i^{-1}\right)
\end{cases}
$$

(1.9)

where $\beta_i^{-1}(\cdot | b_i^{-1})$ is the inverse function of the opening strategy $\beta_i(\cdot | b_i^{-1})$.

The participation threshold for bidder $i$ is $\max(b_i^{-1})$, the previous highest bid, and the exit price is $s_i$, the private synergy. Therefore, a bidder participates only if his synergy exceeds the previous highest bid and he exits truthfully.

The opening strategy is uncertain when the bidder’s synergy is more than the previous highest bid but less than the highest synergy among previous participating bidders, or $\max(b_i^{-1}) \leq s_i < s_i^{\text{max}}$. This uncertainty is because the bidder has no chance of winning in equilibrium, but participating is weakly dominant since surplus remains. The equilibrium payoff is zero; therefore, he is indifferent between a continuum of strategies. His opening bid must be at least the previous highest bid but cannot exceed his private synergy.

When the bidder’s synergy is higher than the synergy of all previous bidders, the opening strategy is uniquely determined. The fact that the bidder has a positive probability of winning in equilibrium leads to a unique best response. Inspecting the opening strategy in equation 1.7 and noting that the signaling incentive $\lambda_i < 1$ and the function $L_i(\cdot)$ is strictly increasing, it follows that the opening bid is strictly increasing in the bidder’s synergy but never exceeds it. In the limit that $\lambda_i$ approaches one, the opening bid equals the synergy.

Furthermore, the opening bid must exceed all previous bidders’ synergies, or $\beta_i \left(s_i | b_i^{-1}\right) > s_i^{\text{max}}$ for all $s_i > s_i^{\text{max}}$. This inequality is because the equilibrium final price will be at least $s_i^{\text{max}}$; therefore, opening with anything less is costless and not credible to signal a synergy higher than $s_i^{\text{max}}$. Empirically, this relation implies that the opening bids after the first bid may also contain a substantial premium, and this feature is important in explaining the observed relatively large second-bid premium in multi-bidder contests in Betton and Eckbo (2000). Section 1.6 will give
more comparison with the data.

Since the opening strategy is strictly increasing and invertible, the market uniquely determines the synergy through the opening bid, as equation 1.9 shows.

Bidder 1’s opening strategy is particularly simple. He will always participate, and the opening strategy is uniquely given by $\beta_1(s_1) = L_1^{-1}(\lambda_1 L_1(s_1))$. I examine two specific cases with simple solutions.

**Example 2.** If there is only one bidder, then $L_1(x) = x - \bar{s}$ and the opening strategy is $\beta_1(s) = \bar{s} + \lambda_1(s - \bar{s})$.

Even when there is only one bidder, jump bidding still results due to the signaling incentive. This feature is different from Fishman (1988), Daniel and Hirshleifer (1997), and Avery (1998), which all predict no jump bidding with a single bidder. This feature helps the model match the observed initial premium as Section 1.6 will discuss.

As the expected number of active bidders increases, the initial bid increases. This increase is due to the fact that a fixed initial bid becomes less credible with more active rivals as a bidder is less likely to win with it; therefore, the initial bid increases in order to maintain credibility. To quantify this effect I explicitly solve for the opening strategy when $F(\cdot)$ is uniform distribution and the number of active bidders is deterministic.

**Example 3.** If all bidders are active with probability 1 and the synergy is distributed uniformly on $[\bar{s}, \bar{s}]$, then $L_1(x) = \frac{1}{n} \frac{(x-\bar{s})^n}{(\bar{s}-\bar{s})^n}$ and the first bidder’s opening strategy in equation 1.10 is

$$\beta_1(s_1) = \bar{s} + (\lambda_1)^{\frac{1}{n}} (s_1 - \bar{s}), \quad (1.10)$$

and the market’s belief in equation 1.9 is, if bidder 1 wins,

$$\theta_1(b_1, p) = \begin{cases} (\lambda_1)^{-\frac{n}{2}} b_1 & \text{if} \quad 0 \leq b_1 \leq (\lambda_1)^{\frac{1}{n}} \\ 1 & \text{if} \quad b_1 > (\lambda_1)^{\frac{1}{n}} \end{cases}. \quad (1.11)$$

With uniform distribution and a deterministic number of active bidders, the initial premium is linear in the synergy and the coefficient is $(\lambda_1)^{\frac{1}{n}}$. Figure 1.2 plots this coefficient as a function of the number of bidders in a special case where bidder 1’s signaling incentive is $\lambda_1 = 5\%$.

As can be seen from the figure, although the signaling incentive is small, the initial bid premium coefficient increases toward one rapidly with the number of bidders due to the power relationship. For example, for $\lambda_1 = 5\%$ and $n = 3$, the initial bid premium coefficient is $37\%$. This means that if there are three bidders and the winner sells only 5% of the joint company after
the auction, the first bidder offers an initial premium of 37% of his synergy. Therefore, even a small signaling incentive can produce a significant bid premium with a few bidders. Consequently, if one wishes to understand offer premia in takeovers where the number of bidders is not too small, it is necessary to take into account any signaling incentives even if they are tiny.

The fact that the winner’s synergy is fully revealed in the pure separating equilibria has two implications, both are different from the pure pooling equilibria to be discussed next. First, the exit price, as in equation 1.8, is truthful and does not depend on \( \lambda_i \). Therefore, the separating equilibria are efficient even if bidders have different signaling incentives.

**Proposition 3.** *The separating equilibria are efficient.*

Another implication exists on the winner’s post-takeover stock price movements. Referring to equations 1.2 and 1.3, the winner’s short- and long-term post-takeover market values are the
same, or more precisely, \( v_2 = v_3 = s_i - p \). Therefore, the winner’s stock price does not move after the takeover. Furthermore, the final price is the synergy of the highest losing bidder. Therefore, the winner’s profit is strictly positive except in events of zero probability, where his synergy ties with the highest losing bidder. Therefore, the probability of a decrease in the winner’s market value after the takeover is zero.

**Lemma 5.** In the pure separating equilibria, with probability one, the winner’s market value strictly increases after the takeover and stays the same afterward, or \( v_3 = v_2 > v_0 \).

Last, I examine how the initial bid premium differs in single- and multi-bidder contests. Two sources contribute in opposite directions. First, an ex-post selection bias exists and a high initial premium is more likely to result in a single-bidder contest because it may exceed the synergies of all other bidders. This effect makes the initial premium in a single-bidder contest slightly higher than in a multi-bidder contest, which agrees with the data. Second, an opposite effect arises if the data are aggregated over samples that differ in the expected number of active bidders. This is because the initial bid premium increases in the expected number of active bidders, and thus a single-bidder contest is more likely to have been drawn from samples with a smaller expected number of active bidders and will tend to have a lower initial premium.

If the data are aggregated over samples with identical distributions in the number of active bidders, that is, if all samples have the same maximum number of active bidders \( n \) and the same active probability \( q_i \) for any \( i \leq n \), then only the ex-post selection bias effect exists. In this case my model predicts that the initial premium in a single-bidder contest is slightly higher than in a multi-bidder contest, the same as observed in the data.

**Proposition 4.** If the data are aggregated over samples with identical distributions in the number of active bidders, the expected initial bid premium in a single-bidder contest is higher than in a multi-bidder contest, that is, \( E[b_1|\text{single-bidder}] > E[b_1|\text{multi-bidder}] \).

### 1.4.2 The Pure Pooling Equilibria

In this section I examine the pure pooling equilibria, where the private synergy is signaled through the final price. As has been mentioned earlier, there is a continuum of pooling equilibria that differ trivially. The main result of this section follows.

**Proposition 5.** The set of pure pooling equilibria is nonempty. For all bidder \( i \) and the previous
opening bids $b_i^{j-1}$, where $b_i^{j-1}$ is on the equilibrium path,

\[(i) : \beta_i (s_i | b_i^{j-1}) = \beta_i (s'_i | b_i^{j-1}) \text{ for all } s_i \text{ and } s'_i \quad (1.12)\]
\[(ii) : \gamma_i (s_i | b_i^{j-1}) = (1 - \lambda_i) s_i + \lambda_i E[s | s \geq s_i] \quad (1.13)\]
\[(iii) : \underline{s} \leq \beta_i (s_i | b_i^{j-1}) \leq \gamma_i^\text{min}, \quad (1.14)\]

where $\gamma_i^\text{min} \equiv \gamma_i (\underline{s} | b_i^{j-1}) = (1 - \lambda_i) \underline{s} + \lambda_i E[s | s \geq \underline{s}]$ is the minimum exit price for $i$ when $s_i = \underline{s}$.

The market’s belief on the equilibrium path is

$$\theta_i = E [s | \gamma_i (s | b_i^{j-1}) \geq p]. \quad (1.15)$$

In the pure pooling equilibria, all active bidders will participate. Equation 1.12 simply says a bidder’s opening bid is independent of the synergy. Since the opening bid contains no information, the bidder uses the exit price to signal his synergy. The exit price is uniquely determined from proposition 1 and is given by equation 1.13. It is independent of the previous opening bids $b_i^{j-1}$ as history contains no information. The exit price exceeds and increases in the private synergy due to the signaling incentive.

The level of the opening bid is not uniquely determined. Qualitatively, this indeterminacy is because the exit prices of all bidders are bounded away from $\underline{s}$, and opening in a neighborhood of $\underline{s}$ incurs no cost for the bidder. On the other hand the opening bid cannot exceed $\gamma_i^\text{min}$, which is the minimum exit price and is greater than $\underline{s}$.

The market does not learn the winner’s exact synergy in the pure pooling equilibria, only its expectation conditional on $p$. The market’s belief increases in $p$ and exceeds the actual synergy $s_i$ if $p$ is high enough. For example, when the bidder wins at his exit price $p = \gamma_i (s_i | b_i^{j-1})$, the market’s belief about his synergy is $\theta_i = E[s | s \geq s_i]$, which exceeds $s_i$. In addition, the exit price in the pure pooling equilibria increases in $\lambda_i$, the signaling incentive.

Consider the following example.

**Example 4.** If the synergy is distributed uniformly on $[\underline{s}, \bar{s}]$, the exit strategy in equation 1.13 is

$$\gamma_i (s) = s + \frac{\lambda_i}{2} (\bar{s} - s),$$

the minimum exit price for bidder $i$ is $\gamma_i^\text{min} = \underline{s} + \frac{\lambda_i}{2} (\bar{s} - \underline{s})$, and the market’s belief in equation 1.15 is

$$\theta_i = \begin{cases} \frac{1-\lambda_i}{2-\lambda_i} \bar{s} + \frac{p}{2-\lambda_i} (\bar{s} - \underline{s}) & \text{if } p \geq \gamma_i^\text{min} \\ \frac{1}{2} (\bar{s} + \underline{s}) & \text{if } p < \gamma_i^\text{min}. \end{cases} \quad (1.16)$$
When the synergy distribution is uniform, the exit price linearly increases in the signaling incentive, and the market’s belief linearly increases in the price.

The increase of the exit price on the signaling incentive implies that the bidder with the highest signaling incentive is more likely to be the winner. Therefore, the pure pooling equilibria are not efficient unless all bidders have the same signaling incentive.

**Proposition 6.** The pure pooling equilibria are efficient if and only if the signaling incentive \( \lambda_i \) is the same for all bidder \( i \).

Because inefficiency leads to future resale, the pure pooling equilibria induce a larger volume of future resale when bidders have different signaling incentives.\(^{14}\) Section 1.6 discusses this effect in more detail.

I now examine the winner’s post-takeover stock price movements. For comparison, recall that \( v_3 = v_2 > v_0 \) in the pure separating equilibria. In the pure pooling equilibria, the above relation is only true in expectation, or \( E[v_3] = E[v_2] > v_0 \).

**Lemma 6.** In the pure pooling equilibria, \( E[v_3] = E[v_2] > v_0 \).

The intuition is that the market’s belief at \( t = 2 \) about the winner’s synergy is its expectation conditional on \( p \). From the law of iterated expectations, \( E[v_3] = E[v_2] \). The reason for \( E[v_3] > v_0 \) is that the winner’s expected profit is strictly positive.

However, since the market does not learn the winner’s exact synergy, the relation in Lemma 6 does not hold for all random outcomes and the winner’s post-takeover stock price is volatile.

**Lemma 7.** The winner’s post-takeover stock price is volatile in the pure pooling equilibria. Or \( \text{Var}[v_3 - v_2] > 0 \).

Bidders stay in the auction past their actual synergies, leading to a possible overpayment. From equations 1.2 and 1.3, there is a positive probability that the winner’s market value decreases after the takeover; that is, \( v_3 < v_0 \) and \( v_2 < v_0 \).

**Lemma 8.** If the final price is more than the winner’s synergy and the synergy is independently revealed to the market at \( t = 2 \), the winner’s market value is less after the takeover. Or more precisely, \( v_3 = v_2 < v_0 \).

---

\(^{14}\)Note that the preceding analysis has no resale built in the model. For consistency, we can show that the statements made on resale in the pure separating and pure pooling equilibria still go through in an enhanced model with resale. Heuristically, since the pure separating equilibria are efficient in a model without resale, there will still be no resale when resale is allowed. In the pure pooling equilibria, we show resale necessarily occurs by contradiction. Suppose not, then the enhanced model reduces to a model without resale. But in such a model, the pure pooling equilibria are generically inefficient, contradicting the assumption that resale never occurs.
The intuition for the winner’s post-takeover discount is that a bidder is willing to bid above the actual synergy in the hope that the market at \( t = 2 \) will not learn his exact synergy and will have to estimate it based on the final price, and the estimate will exceed the actual synergy when the final price is high. However, if the winner’s synergy is independently revealed at \( t = 2 \), the winner has paid too much.

I summarize the winner’s stock price movements and compare with the pure separating equilibria below.

**Proposition 7.** In the pure pooling equilibria, the winner’s post-takeover stock price volatility is greater than zero, and a positive probability arises for a decrease in the winner’s stock price after the takeover. Or \( \text{var}(v_3 - v_2) > 0 \), \( \text{prob}(v_2 < v_0) > 0 \), and \( \text{prob}(v_3 < v_0) > 0 \). By contrast, none of the above is true in the pure separating equilibria.

Empirically, the pure pooling and pure separating equilibria can be distinguished by the size of the initial bid. The expected value of the initial bid in any pure separating equilibrium is more than \( \gamma_1^\text{min} \), the upper bound of the initial bid in any pure pooling equilibrium.

**Proposition 8.** In any pure separating equilibrium, the expected initial bid \( E[\beta_1 (s_1)] \geq \gamma_1^\text{min} \) and the inequality is strict when the maximum number of bidders, \( n \), is greater than 1. Therefore, the expected value of the first bid is higher in any pure separating equilibrium than in any pure pooling equilibrium.

### 1.5 Bidders’ and the Target’s Profits

Signaling is costly for bidders. In the pure separating equilibria, the cost comes from the opening premium bidders offer, whereas in the pure pooling equilibria, the cost comes from the possibility of overpayment. In this section I quantify how signaling incentives affect the profits of bidders and the target. I will focus only on efficient equilibria, which include the pure separating equilibria and the pure pooling equilibria when all bidders have the same signaling incentives. The primary reason for focusing on the efficient equilibria is to simplify the analysis. Social welfare is maximized in any efficient equilibrium and is determined by the distribution of the highest synergy among active bidders. Therefore, social welfare is the same among all efficient equilibria. Since the market breaks even, the sum of profits from the target and all bidders is the same among efficient equilibria, and the bidders’ losses directly translate into the target’s gain.

Technically, a bidder’s expected profit can be derived through incentive compatibility and the derivation depends on a boundary value: the bidder’s expected profit conditional on his synergy.
being at the lower bound \( s \). This boundary value needs to be determined before proceeding. Note this value is zero in most standard auctions because bidders of the lowest type receive no rents in general. But this value may be positive in my model due to a subtlety that the number of active bidders is random and sometimes only one active bidder exists and faces no competition. However, under certain loose conditions, this value will be zero.

**Lemma 9.** In any efficient equilibrium satisfying assumptions 1 through 6, any bidder’s expected profit is zero conditional on having the lowest synergy \( s \) under either of the two following conditions: (i) the probability of being active \( q_i = 1 \) for some \( i > 1 \) such that there are always two or more active bidders, or (ii) the equilibrium is pure separating.

Condition (i) ensures zero profit for a bidder with synergy \( s \) by means of competition when at least two active bidders exist. Condition (ii) ensures zero profit for a bidder with synergy \( s \) because his synergy is fully revealed in the pure separating equilibria; therefore, even if he faces no competition, he still cannot garner a positive expected profit.

I next quantify the expected profits of bidders and the target by assuming either of the two conditions in Lemma 9 holds. To proceed, define \( G_0 (s) \equiv \Pi_{j=1}^n (1 - q_j + q_j F (s)) \), which is the distribution of the highest synergy among all active bidders. This distribution determines the increase in the social welfare due to the takeover. I also define \( G_{-i} (s) \equiv \Pi_{j=1,j\neq i}^n (1 - q_j + q_j F (s)) \), which is the distribution of the highest synergy among all active bidders other than \( i \). This distribution is the winning probability for bidder \( i \) in an efficient equilibrium.

**Proposition 9.** In any efficient equilibrium satisfying assumptions 1 through 6, if either of the two conditions in Lemma 9 holds, the expected profit of bidder \( i \) is

\[
\pi_i = q_i (1 - \lambda_i) \int_{\tilde{s}}^{s} dF (s) \int_{\tilde{s}}^{s} G_{-i} (y) dy
\]

and the expected profit of the target is \( \int_{\tilde{s}}^{s} s dG_0 (s) - \Sigma_{i=1}^n \pi_i \).

Under the proposition a bidder’s profit is the same in all efficient equilibria, as is the target’s profit. Revenue equivalence obtains because synergies are private and independent. On the other hand, the bidder’s profit is reduced by a factor \( (1 - \lambda_i) \) because a bidder receives rents on the private synergy but not on the portion that is signaled and made public. Therefore, signaling incentives reduce the profit of a bidder and increase the profit of the target.

Note that a bidder’s expected profit depends on \( G_{-i} (y) \), which is independent of the order of arrival in bidding. This lack of dependence justifies the earlier assumption of exogenous order for the bidders because the bidders are indifferent.
Last, I briefly discuss inefficient equilibria, which are the pooling equilibria where bidders have different signaling incentives. The sum of profits of the target and all bidders is less compared with the efficient equilibria. But in terms of the profit of the target alone and the profit of each individual bidder, the comparison is complicated and is outside of the scope of this paper. Qualitatively, since a higher $\lambda_i$ leads to a higher exit price, the bidder with the highest $\lambda_i$ is more likely to win in an inefficient equilibrium than in an efficient one and he receives an increased profit. Similarly, the profit of the bidder with the smallest $\lambda_i$ decreases in an inefficient equilibrium. Therefore, an inefficient equilibrium preferentially benefits bidders with larger signaling incentives.

1.6 Empirical Implications

In the first part of this section, I calibrate the pure separating equilibria in a parametric example and match the observed bidding characteristics. In the second part of this section, I derive general empirical implications from the model.

1.6.1 Calibrating the Pure Separating Equilibria and Matching the Bidding Characteristics

As has been discussed earlier, the model explains the observed bidding characteristics in Betton and Eckbo (2000) if we assume the data are aggregated over samples with identical distribution in the number of active bidders. In this part of the section, I make this assumption and use the pure separating equilibria from my model to match the observations. Using the pure separating equilibria instead of the pure pooling equilibria gives my model the best chance in matching the data because the initial bid premium is lower in the pure pooling equilibria.

My model has predictions for the following six variables observed in Betton and Eckbo (2000):

- $f_{\text{sin}}$: The fraction of single-bidder contests.
- $i_{p,\text{sin}} \equiv \frac{b - V_t}{V_t}$: Initial bid premium in single-bidder contests, where $V_t$ is the target value before the takeover.
- $i_{p,\text{mul}}$: Initial bid premium in multi-bidder contests, similarly defined as in single-bidder contests.
- $f_{r\text{w}}$: The fraction of multi-bidder contests where a rival bidder wins.
Variables & $f_{\text{sin}}$ & $i p_{\text{sin}}$ & $i p_{\text{mul}}$ & $f_{rw}$ & $fp$ & $sp$
Data & 78% & 51% & 45% & 81% & 74% & 65%
Model & 78% & 51% & 45% & 77% & 71% & [67%, 70%]

Table 1.1: Bidding Characteristics from the Data and the Model.

- $fp \equiv \frac{p-V_t}{V_t}$: Final offer premium in multi-bidder contests.
- $sp \equiv \frac{b_{2,\text{par}}-V_t}{V_t}$: Second-bid premium in multi-bidder contests, where $b_{2,\text{par}}$ is the bid of the second participating bidder.

The observed values of these variables are listed in the first row of table 1.1.\(^\text{15}\)

I normalize the target value $V_t = 1$ in the model without loss of generality. I also set the maximum number of bidders $n = 5$ based on observations in Betton and Eckbo (2000). Furthermore, I assume the synergy distribution is uniform. I also assume all bidders have the same signaling incentive $\lambda$. Recall that the first bidder is always active. I now assume all other bidders have the same active probability $q$. Given these assumptions, the model has a total of four free parameters: $s$, $\bar{s}$, $\lambda$, and $q$.

The four parameters are chosen to exactly match the first three variables ($f_{\text{sin}}$, $i p_{\text{sin}}$, and $i p_{\text{mul}}$) and to minimize the sum of square errors from the last three ($f_{rw}$, $fp$, and $sp$).\(^\text{16}\) The values of the parameters are $s = 0.19$, $\bar{s} = 1.10$, $\lambda = 0.65$, and $q = 0.093$. The second row of table 1.1 lists the model predictions under these parameters.

The reason the model matches the relatively small difference in the initial bid premium is because a high initial bid is more likely to exceed the actual synergies of remaining bidders and lead to a single-bidder contest. In addition, the model predicts a rival bidder wins in 77% of the multi-bidder contests. This predicted fraction is not 100% because a rival bidder participates as long as surplus remains, even if he has no chance of winning. For comparison, the observed fraction is 81%. Furthermore, the model also matches the observed higher final offer premium in multi-bidder contests because the first bidder’s synergy is not necessarily small in multi-bidder contests and thus will remain in the auction until a higher price obtains.

When the second bidder’s synergy is less than the first bidder’s synergy but more than the initial bid, the second bid in my model is indeterministic and the model generates a range for the

\(^{15}\)The portion of the data where the first bidder raises his own bid after the initial bid is excluded. Betton and Eckbo (2000) directly report values of the first five variables. The last variable is inferred from their reported second jump revision ratio.

\(^{16}\)The model predicts a range for the second-bid premium due to the second-bid indeterminacy as mentioned earlier. The error is defined to be the closest distance between its observed value and any point in the predicted range. For example, if the observed value is within the range, the error is zero.
second-bid premium. When the second bidder’s synergy exceeds the first bidder’s synergy, the second bid also exceeds the first bidder’s synergy in order to be a credible signal. This feature gives rise to a relatively large second-bid premium as required by the data. The observed value is only slightly outside the predicted range.

Overall, the six variables are well matched with four parameters.

The assumption that the active probability is the same for all bidders $i > 1$ is made for simplicity. However, other parameterization on the active probabilities (e.g., geometrically declining probabilities) has also been examined and yields essentially the same results because once the observed fraction of single-bidder contests is matched, the results are insensitive to details of bidders’ active probabilities.

The parameter value for the signaling incentive $\lambda$ is 0.65. This value would correspond, for example, to a situation where the winner sells the entire target after winning and the market has a 35% probability of independently learning the synergy. This value of $\lambda$ is large for two reasons. First, the observed large value for $f_{\text{sin}}$ (78%) implies that the expected number of active bidders is only slightly more than one. In order to produce a sizable initial bid premium, a large signaling incentive is needed. Second, this value of $\lambda$ generates the optimal match for all six variables. If the quality of the match is relaxed, a smaller value of $\lambda$ can be obtained.

The size of the parameter $\lambda$ suggests that the model is best able to explain the comparative statics of the observed bidding premia, not their exact levels.

### 1.6.2 General Empirical Implications

This paper contains a number of novel empirical predictions and they depend on signaling incentives. A set of proxies exist for signaling incentives. Some of them are better than others because they are not contaminated by endogeneity problems. These proxies include the cash equivalent of equity payment, collar clause in the payment agreement, or generally when bidders bid with equity. In addition, signaling incentives also arise when the bidder CEO’s compensation depends on the bidder’s short-term performance, for example if he is paid with options instead of cash. Other proxies include post-takeover observations that the winner sells all or part of the target or issue equity to finance the purchase. These proxies are potentially contaminated because the decision to sell or issue equity may be made ex-post. However, they are still useful in that, conditional on observing them ex-post, bidders are more likely to have had incentives to signal high synergies in the bidding stage.

Signaling incentives have implications on the initial bid premium. Without signaling incentives the initial bid is simply the minimum required bid: the reserve price. With signaling
incentives bidders offer higher initial premium in the pure separating equilibria. Although the model does not predict the probability with which the pure separating equilibria occur, as long as the probability is greater than zero, the expected initial bid premium will be higher with signaling incentives.

**Corollary 10.** *Everything else being equal, conditional on observing the proxies for signaling incentives, the initial bid premium is higher in expectation.*

Signaling incentives also lead to the pure pooling equilibria distinguishable from the pure separating equilibria by a lower initial bid premium. If we assume data are aggregated over samples, and that part of them are in pure separating equilibria and the rest are in pure pooling equilibria, then conditional on observing a larger initial premium, the corresponding sample is more likely to be in a pure separating equilibria. Since the two types of equilibria have different properties on the allocative efficiency and the winner’s post-takeover stock price performance and volatility, a number of novel empirical correlations exist.\(^{17}\)

Because inefficiency leads to observable future resale, the fact that the pooling equilibria are inefficient when bidders differ in the signaling incentives has implications on resale activities.

**Corollary 11.** *The expected volume of future resale decreases in the initial premium and increases in the heterogeneity of bidders’ signaling incentives.*

For example, one would expect to observe larger future resale if some of the bidders bid with cash and others bid with equity or if some bidders’ CEOs are paid with cash and others are paid with options.

The amount of information released is smallest in the pure pooling equilibria; therefore, the winner’s post-takeover stock price volatility in the pure pooling equilibria is larger than in the pure separating equilibria.

**Corollary 12.** *Conditional on a higher initial premium, the winner’s post-takeover stock price is less volatile in expectation.*

In addition, the pure pooling equilibria result in possible overpayment, leading to a possibility of a decrease in the winner’s market value after the takeover.

**Corollary 13.** *Conditional on a higher initial premium and a smaller signaling incentive, after the takeover the winner’s stock price is less likely to decrease over that before the takeover.*

\(^{17}\)The empirical correlations are robust even if we assume part of the sample may be in the hybrid equilibria. The robustness is because the hybrid equilibria have properties in between the pure separating and pure pooling equilibria and thus do not affect the correlations.
As discussed earlier, this prediction may be counter-intuitive as casual intuition may suggest a high initial bid is more likely to result in an overpayment. According to my model, however, precisely because bidders have signaled their synergies through a high initial bid, they have no need to signal again with an overpayment. Therefore, conditional on observing a larger initial premium, overpayment is less likely.

Furthermore, signaling incentives reduce the bidders’ profits and increase the target’s profit. This result has both positive and normative implications. Positively, we have the following:

**Corollary 14.** The expected change in the winner’s market value after the takeover is less positive for larger signaling incentives. The reverse is true for the target.

Normatively, the bidders and the target can benefit if they can influence the signaling incentives. A bidder’s profit is reduced if he has to signal the synergy to the market. Therefore, the preferred means of financing for a bidder is to use internal capital or to issue debt, as opposed to issuing equity. As has been discussed earlier, this preference is the same as in the pecking-order theory, but for different reasons. In the pecking-order theory, the choice of financing is endogenous and equity issuance is itself a signal that the bidder is overvalued. On the other hand, the choice of financing in my model is exogenous, and equity issuance reduces the bidder’s profit because it forces the bidder to signal his private value in the preceding auction. This preference may help explain the prevalence of LBOs because financing with debt is preferable to financing with equity.

On the other hand, the target benefits when bidders have signaling incentives. One way the target can induce signaling incentives is by requesting the cash equivalent of equity payment as opposed to cash payment, and my model suggests the target benefits by doing so. This result is also a new one from this paper and is robust even in the Myers-Majluf framework with asymmetric information. The robustness is because the cash equivalent of equity offer guarantees the cash value of the offer and thus is independent of any asymmetric information.

### 1.7 Conclusions

This paper explores why bidders offer high initial premia in takeovers, how the market reacts to such offers, and what consequences the initial premia have on the allocative efficiency and the winner’s post-takeover stock price performance and volatility. The paper uses a parsimonious model in which bidders with private synergy values participate in an ascending price auction and have incentives to signal high synergies to the market.
Bidders have two ways to signal high synergies. One is by offering a premium in the opening bid. The opening bid strictly increases in the bidder’s synergy, resulting in the separating equilibria. The separating equilibria correspond to the maximum amount of information release as the opening bid fully reveals the private information. Consequently, bidders exit at their actual synergies, the allocation is efficient, and the winner’s post-takeover stock price volatility is smaller.

The separating equilibria provide a possible explanation for the stylized facts on the observed opening premia in takeovers. In one parametric example the model predictions accord well with the observed takeover bidding characteristics.

Another way bidders signal a high synergy is with a high final price, resulting in the pooling equilibria. The opening bid is a constant and low, and the market infers the winner’s synergy through the final price. Because the opening bid conveys no information and the final price does not reveal the exact synergy, the pooling equilibria correspond to the smallest amount of information release. Thus the allocation is generically inefficient, leading to larger future resale. The winner’s post-takeover stock price volatility is greater.

A high price signals a high synergy in the pooling equilibria. Therefore, bidders stay in the auction past their actual synergies, leading to possible overpayment. A positive probability arises for a decrease in the winner’s stock price after the takeover.

The initial bid premium in any separating equilibrium is higher than in any pooling equilibrium. Therefore, the model predicts novel correlations among the initial bid premium, the winner’s post-takeover stock price performance, volatility, and future resale. For example, conditional on a higher initial bid premium, the winner’s stock price after the takeover is less likely to decrease over that before the takeover. It is also less volatile and future resale is smaller.

Signaling incentives increase the profit of the target and decrease the profit of a bidder. The paper suggests the target (or a bidder) can benefit by influencing the signaling incentives. The target should request contingent equity payments (e.g., the cash equivalent of equity payment) as opposed to cash payments. For the bidder using internal capital or issuing debt is best, as opposed to issuing equity, in order to finance the purchase.
Bibliography


1.8 Appendix A: Proofs

Proof of Proposition 1

Let \( \pi (b, s_i) \) denote the expected profit of bidder \( i \) with signal \( s_i \) when he opens with the equilibrium strategy \( \beta_i(s_i|b^{i-1}) \) and follows an arbitrary exit strategy \( b \), where \( b \geq \beta_i(s_i|b^{i-1}) \). Let \( k(\cdot) \) denote the density distribution of the highest exit price among the other active bidders. Then

\[
\pi (b, s_i) = \int_{\beta_i(s_i|b^{i-1})}^{b} [\lambda_i \theta_i (\beta_i(s_i|b^{i-1}), p|b^{i-1}) + (1 - \lambda_i) s_i - p] k(p) \, dp.
\]

Since the equilibrium exit strategy is a best response, it is

\[
\gamma_i(s_i|b^{i-1}) = \arg \max_{b \geq \beta_i(s_i|b^{i-1})} \pi (b, s_i).
\]

We first establish the following two claims.

Claim 1: \( \gamma_i(s_i|b^{i-1}) = \lambda_i \theta_i (\beta_i(s_i|b^{i-1}), \gamma_i(s_i|b^{i-1})|b^{i-1}) + (1 - \lambda_i) s_i. \)

To prove claim 1 we first define \( \Delta(s_i) \) to be the difference between the RHS and LHS for notational convenience, or

\[
\Delta(s_i) \equiv \lambda_i \theta_i (\beta_i(s_i|b^{i-1}), \gamma_i(s_i|b^{i-1})|b^{i-1}) + (1 - \lambda_i) s_i - \gamma_i(s_i|b^{i-1}).
\]

We show \( \Delta(s_i) = 0 \) by contradiction. First suppose that \( \Delta(s_i) > 0 \). Since \( \theta_i (\beta_i(s_i|b^{i-1}), p|b^{i-1}) \) is nondecreasing in \( p \), it is straightforward to show that \( \pi (\gamma_i(s_i|b^{i-1}) + \Delta(s_i), s_i) \geq \pi (\gamma_i(s_i|b^{i-1}), s_i) \).

Furthermore, the inequality sign will hold strictly in situations where \( k(p) \) is positive on part of the interval \((\gamma_i(s_i|b^{i-1}) + \Delta(s_i), \gamma_i(s_i|b^{i-1}) + \Delta(s_i)) \). This possibility of strict inequality means that exiting at \( \gamma_i(s_i|b^{i-1}) \) is weakly dominated by exiting at \( \gamma_i(s_i|b^{i-1}) + \Delta(s_i) \), contradicting assumption 2.

Next suppose \( \Delta(s_i) < 0 \). We discuss two situations: (i) if \( \gamma_i(s_i|b^{i-1}) > \beta_i(s_i|b^{i-1}) \). Then one can show that exiting at \( \gamma_i(s_i|b^{i-1}) \) is weakly dominated by exiting at

\[
\max \{ \gamma_i(s_i|b^{i-1}) + \Delta(s_i), \beta_i(s_i|b^{i-1}) \}
\]

and this contradicts assumption 2. (ii) if \( \gamma_i(s_i|b^{i-1}) = \beta_i(s_i|b^{i-1}) \), then the bidder’s payoff will be negative if he plays the equilibrium strategy and wins, thus the equilibrium strategy is weakly dominated by not participating. This contradicts assumption 2.

Therefore claim 1 is true.

Claim 2: For any \( s'' > s' \) such that \( \beta_i(s''|b^{i-1}) = \beta_i(s'|b^{i-1}) \), we have \( \gamma_i(s''|b^{i-1}) > \gamma_i(s'|b^{i-1}) \).

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To prove claim 2, note that \( \gamma_i (s''|b_i^{j-1}) \geq \gamma_i (s'|b_i^{j-1}) \) from assumption 3. Suppose the equality sign were to hold, then we could plug \( s'' \) and \( s' \) respectively into claim 1 and take the difference to get \( s'' = s' \), which contradicts the earlier assumption that \( s'' > s' \). Thus claim 2 is established.

Plugging the market’s belief in equation 1.1 into claim 1 yields

\[
\gamma_i (s_i|b_i^{j-1}) = \lambda_i \mathbb{E}[s_i\beta_i (s_i|b_i^{j-1})] = \beta_i (s_i|b_i^{j-1}), \quad \gamma_i (s_i|b_i^{j-1}) \geq \gamma_i (s_i|b_i^{j-1})] + (1 - \lambda_i) s_i.
\]

By claim 2 the above yields the proposition.

**Proof of Corollary 1**

First notice that \( \gamma_i (s_i|b_i^{j-1}) \geq s_i \) follows readily from proposition 1.

To prove the second part of the corollary, define \( s^* \equiv \max \{ s : \beta_i (s_i|b_i^{j-1}) = \beta_i (s_i|b_i^{j-1}) \} \). We discuss two cases. (i) If \( s_i = s^* \), proposition 1 yields \( \gamma_i (s_i|b_i^{j-1}) = s_i \). (ii) If \( s_i \neq s^* \), then \( s_i < s^* \). Since \( \beta_i (s_i|b_i^{j-1}) \) is nondecreasing, we have that \( \beta_i (s_i|b_i^{j-1}) = \beta_i (s_i|b_i^{j-1}) \) for all \( s' \in (s_i, s^*) \), and then proposition 1 yields \( \gamma_i (s_i|b_i^{j-1}) > s_i \). This establishes the second part of the corollary.

**Proof of Lemma 2**

Since the solution to \( \beta_i (s_i|b_i^{j-1}) = b_i \) and \( \gamma_i (s_i|b_i^{j-1}) \geq p \) is an interval, denote by \( s' \) and \( s'' \) the lower and upper limits. From equation 1.1 we have \( \theta_i \equiv \mathbb{E}[s|s' \leq s \leq s''] \). On the other hand, we can show that \( \gamma_i (s_i|b_i^{j-1}) \equiv \mathbb{E}[s|s' \leq s \leq s''] \) from proposition 1. Therefore, \( \gamma_i (s_i|b_i^{j-1}) \equiv \theta_i \). In addition we have \( \max \{ b_i^{j-1} \} \leq b_i = \beta_i (s_i|b_i^{j-1}) \leq \gamma_i (s_i|b_i^{j-1}) \). Combining this with the previous inequality, the lemma follows.

**Proof of Lemma 3**

For notational convenience we use \( \alpha (b_i^{j-1}) \) to denote the participating threshold of bidder \( i \).

Claim 1: Bidder \( i \)'s exit strategy \( \gamma_1 (\cdot) \) is strictly increasing.

To prove the claim, pick any \( s'' > s' \). Since \( \beta_1 (\cdot) \) is nondecreasing, we discuss two cases. (i) if \( \beta_1 (s'') = \beta_1 (s') \), then from proposition 1 we readily have \( \gamma_1 (s'') > \gamma_1 (s') \). (ii) if \( \beta_1 (s'') > \beta_1 (s') \), then from proposition 1 we have \( \gamma_1 (s') < \lambda_1 s'' + (1 - \lambda_1) s' < s'' \). On the other hand, we also have \( \gamma_1 (s''') \geq s''' \). Therefore, \( \gamma_1 (s''') > \gamma_1 (s') \).

Claim 2: For any \( i \), any \( b_i^{j-1} \) and any \( s^* \) such that \( s^* \geq \alpha (b_i^{j-1}) \), if we define \( s'_1 \equiv \min \{ s : \beta_i (s) = \beta_i (s^*) \} \) and \( s''_i \equiv \max \{ s : \beta_i (s) = \beta_i (s^*) \} \), and additionally

\[
s'_1 \equiv \min \{ s : \beta_i (s_i|b_i^{j-1}) = \beta_i (s^*|b_i^{j-1}) \} \quad \text{and} \quad s''_i \equiv \max \{ s : \beta_i (s_i|b_i^{j-1}) = \beta_i (s^*|b_i^{j-1}) \},
\]

we have \( s'_1 = s'_i \) and \( s''_i = s''_i \).

To prove the claim note that \( s'_1 \leq s^* \leq s''_1 \) by construction. Since \( \beta_1 (s'_1|b_i^{j-1}) = \beta_1 (s^*|b_i^{j-1}) = \beta_1 (s'_1|b_i^{j-1}) \), by assumption 5 and the fact that \( s^* \geq \alpha (b_i^{j-1}) \), we have \( \beta_i (s'_1|b_i^{j-1}) = \beta_i (s^*|b_i^{j-1}) = \beta_1 (s'_1|b_i^{j-1}) \). This implies that \( s'_1 \geq s'_i \) and \( s''_i \leq s''_i \).

Suppose \( s'_1 > s'_i \). Then we have \( \beta_1 (s'_i) < \beta_1 (s'_1) = \beta_1 (s_i) \). By assumption 5 again we
have $\beta_i(s'_i|\bar{b}^{i-1}) < \beta_i(s_i|\bar{b}^{i-1})$, which is a contradiction. Therefore, $s'_1 = s'_i$. Similarly, we have $s''_1 = s''_i$. This establishes claim 2.

Claim 3: For any $i$, $\bar{b}_i$ and any $s^*$, if $s^* \geq \alpha(\bar{b}^{i-1})$, then $\gamma_i(s^*|\bar{b}^{i-1}) = \gamma_1(s^*)$.

The claim follows readily from claim 2 and proposition 1.

Claim 4: $\alpha(\bar{b}^{i-1}) \leq \max\{\bar{b}_i\}$ for all $i$, $\bar{b}_i$.

To prove the claim, suppose bidder $i$ has a synergy $s_i > \max\{\bar{b}_i\}$. If he opens with $\max\{\bar{b}_i\}$ and immediately exits, his profit upon winning will be positive from assumption 6, and therefore participation weakly dominates nonparticipation. This establishes the claim.

Claim 5: for any $i$, denote by $j$ the previous participating bidder immediately before him. Then $\alpha_i(\bar{b}_i) \leq s_j$ such that an active bidder $i$ with synergy greater or equal to $s_j$ will always participate.

To prove the claim, define $s''_j \equiv \max\{s: \beta_j(s|\bar{b}_j) = \beta_j(s_j|\bar{b}_j)\}$. Then from proposition 1, we have $\gamma_j(s_j|\bar{b}_j) \leq s''_j$. Due to the constraint that $\beta_j(s_j|\bar{b}_j) \leq \gamma_j(s_j|\bar{b}_j)$, we have $\beta_j(s_j|\bar{b}_j) \leq s''_j$. Using claim 4 we then have

$$\alpha_i(\bar{b}_i) \leq \max\{\bar{b}_i\} = \beta_j(s_j|\bar{b}_j) \leq s''_j. \quad (1.17)$$

On the other hand, since $\beta_j(s_j|\bar{b}_j) = \beta_j(s''_j|\bar{b}_j) > -1$, we have from assumption 5 that $\beta_1(s_j) = \beta_1(s''_j)$, which by assumption 5 again yields: $\beta_i(s_j|\bar{b}_j) = \beta_i(s''_j|\bar{b}_j)$. We have from equation 1.17 that $\beta_i(s''_j|\bar{b}_j) > -1$, thus $\beta_i(s_j|\bar{b}_j) > -1$. This establishes claim 5.

Claim 6: Among all active bidders the one with the highest synergy will participate.

The claim can be easily proved by contradiction using claim 5.

The proposition follows readily from claims 1, 3, and 6.

**Proof of Lemma 4**

First note that $\beta_1(\cdot)$ is continuous in pure separating or pure pooling equilibria. Next we show that $\beta_1(\cdot)$ is discontinuous in any other equilibrium. Using the concept of partitions discussed in Section 1.9, any other equilibrium always contains two neighboring partitions that are not both separating. Let $(s_a, s_b)$ and $(s_b, s_c)$ denote the two partitions. Define $\beta_l \equiv \beta_1(s_b - 0)$ and $\beta_r \equiv \beta_1(s_b + 0)$ to be the left- and right-hand limits of $\beta_1(\cdot)$ at $s_b$. Suppose $\beta_l = \beta_r$. We establish a contradiction in the following three cases.

(i) Both partitions are pooling. Then $\beta_l = \beta_r$ implies that the two partitions can be combined into a single one, and this is a contradiction.

(ii) If the first partition is separating and the second is pooling. It is easy to show that the opening strategy must be continuous within a separating partition; therefore, $\beta_1(\cdot)$ is continuous on $(s_a, s_b)$. Note also that $\beta_l < s_b$. Now consider bidder 1 with synergy $s_1 \in (s_a, s_b)$ and $s_1 > \beta_l$. 40
In equilibrium he opens with $\beta_1 (s_1)$ and exits at $s_1$, and his corresponding profit is

$$\pi^{\text{equi}} (s_1) = K (s_1) s_1 - K (\beta_1 (s_1)) \beta_1 (s_1) - \int_{\beta_1 (s_1)}^{s_1} b d K (b),$$

where $K (\cdot)$ is the distribution of the highest exit bid among all active rivals. Now suppose he deviates by opening with $\beta^r = \beta^l$, still exits at $s_i$, and all other bidders follow the equilibrium strategy. His profit is then

$$\pi^{\text{deviate}} (s_1) = K (s_1) \{(1 - \lambda_1) s_1 + \lambda_1 E [s | s_a < s < s_b]\} - K (\beta^l) \beta^l - \int_{\beta^l}^{s_1} b d K (b).$$

Taking the difference we get

$$\pi^{\text{deviate}} (s_1) - \pi^{\text{equi}} (s_1) = K (s_1) \lambda_1 E [s - s_1 | s_a < s < s_b] - K (\beta^l) \beta^l + K (\beta_1 (s_1)) \beta_1 (s_1) - \int_{\beta^l}^{\beta_1 (s_1)} b d K (b).$$

If we take the left-hand limit that $s_1 \to s_b - 0$, the first term in the above expression is strictly positive, whereas all remaining terms vanish such that $\pi^{\text{deviate}} (s_1) - \pi^{\text{equi}} (s_1) > 0$, contradicting the fact that the equilibrium strategy is the best response.

(iii) If the first partition is pooling and the second is separating, the situation is similar.

Thus we have proved the proposition.
have \( s^* \geq s_i^{\text{max}} \). If \( s^* = s_i^{\text{max}} \), the claim is self-evident. Now suppose \( s^* > s_i^{\text{max}} \) instead. The implication is that \( \beta_i \left( s_i \left| b_i^{-1} \right. \right) < s_i^{\text{max}} \) for all \( s_i < s^* \). Consider some \( s^{**} \) and \( s^{***} \) such that \( s^* > s^{**} > s^{***} > s_i^{\text{max}} \) by assumption. Note the final price will be at least \( s_i^{\text{max}} \) in equilibrium, and opening with a bid of \( \beta_i \left( s^{**} \left| b_i^{-1} \right. \right) \) is at no cost. Therefore, it is strictly better for a bidder with \( s^{***} \) to deviate by opening with \( \beta_i \left( s^{**} \left| b_i^{-1} \right. \right) \) instead (and still exit at \( s^{***} \)) because doing so would signal a higher synergy. Thus this is a contradiction such that having \( s^* > s_i^{\text{max}} \) is impossible. This establishes the claim.

Claim 4: \( \lim_{s_i \to s_i^{\text{max}}} \beta_i \left( s_i \left| b_i^{-1} \right. \right) = s_i^{\text{max}} \).

Taking limits on claims 1 and 3 respectively yields the claim.

The first and second parts of equation 1.7 follow from claims 2 and 1 respectively. To show the last part, consider \( i \)'s best response with \( s_i \geq s_i^{\text{max}} \) when all others employ the equilibrium strategy. Let \( Y \) denote the highest synergy among all active bidders other than \( i \). Note that the highest synergy among the previous \( (i - 1) \) bidders is \( s_i^{\text{max}} \). Therefore, we have \( Y \geq s_i^{\text{max}} \) and its distribution on \([s_i^{\text{max}}, \bar{s}]\) is given by \( G_i(\cdot) \). Let \( \pi(s', y, s_i) \) denote \( i \)'s expected profit when \( i \) opens with \( \beta_i \left( s' \left| b_i^{-1} \right. \right) \) for some \( s' \in [s_i^{\text{max}}, \bar{s}] \) and exits at \( y \geq \beta_i \left( s' \left| b_i^{-1} \right. \right) \). Since all other bidders' exit strategies are truthful by equation 1.8, \( i \) wins if and only if \( y > Y \). Define \( \nu(s', s_i) \equiv (1 - \lambda_i) s_i + \lambda_i s' \) for notational convenience. We then have

\[
\pi(s', y, s_i) = G_i \left( \beta_i \left( s' \left| b_i^{-1} \right. \right) \right) \left( v(s', s_i) - \beta_i \left( s' \left| b_i^{-1} \right. \right) \right) + \int_{\beta_i \left( s' \left| b_i^{-1} \right. \right)}^{y} \left( v(s', s_i) - Y_1 \right) dG_i(Y_1) \\
= G_i(y) v(s', s_i) - G_i \left( \beta_i \left( s' \left| b_i^{-1} \right. \right) \right) \beta_i \left( s' \left| b_i^{-1} \right. \right) - \int_{\beta_i \left( s' \left| b_i^{-1} \right. \right)}^{y} Y_1 dG_i(Y_1)
\]

Since the best response is \( s' = s_i \) and \( y = s_i \), we have from FOC

\[
\frac{\partial \pi(s', y, s_i)}{\partial s'} \big|_{s'=s_i, y=s_i} = \\
\lambda_i G_i(s_i) - g_i \left( \beta_i \left( s_i \left| b_i^{-1} \right. \right) \right) \frac{d\beta_i \left( s_i \left| b_i^{-1} \right. \right)}{ds_i} - G_i \left( \beta_i \left( s_i \left| b_i^{-1} \right. \right) \right) \frac{d\beta_i \left( s_i \left| b_i^{-1} \right. \right)}{ds_i} + g_i \left( \beta_i \left( s_i \left| b_i^{-1} \right. \right) \right) \beta_i \left( s_i \left| b_i^{-1} \right. \right) \frac{d\beta_i \left( s_i \left| b_i^{-1} \right. \right)}{ds_i} \\
= \lambda_i G_i(s_i) - G_i \left( \beta_i \left( s_i \left| b_i^{-1} \right. \right) \right) \frac{d\beta_i \left( s_i \left| b_i^{-1} \right. \right)}{ds_i} \\
= 0.
\]
The above can be rearranged into

$$
\lambda_i G_i (s_i) \, ds_i = G_i \left( \beta_i (s_i | \hat{b}^{i-1}) \right) \, d \beta_i (s_i | \hat{b}^{i-1}).
$$

(1.20)

When integrated from both sides and making use of the boundary condition $\beta_i (s_i^{\max} | \hat{b}^{i-1}) = s_i^{\max}$ from claim 4, we have the last part of equation 1.7.

Finally, equation 1.9 comes from equation 1.4 and the assumption that the market’s belief is nondecreasing.

Next we prove the first part of the proposition by showing that the following constitute a pure separating equilibrium: the market’s belief in equation 1.9 and bidders’ opening and exit strategies in equations 1.7 and 1.8, where the second part of equation 1.7 takes the specific form:

$$
\beta_i (s_i | \hat{b}^{i-1}) = s_i \text{ if } \max (\hat{b}^{i-1}) \leq s_i < s_i^{\max}.
$$

(1.21)

To proceed, first note that equation 1.9 satisfies the consistency requirement in equation 1.4. Next we consider $i$’s best response assuming all others employ the strategies prescribed above. We first look at the case when $s_i \geq s_i^{\max}$. Since $\beta_i (s_i^{\max} | \hat{b}^{i-1}) = s_i^{\max}$ by construction and the final price $p \geq s_i^{\max}$, opening with $\hat{b}_i < s_i^{\max}$ is weakly dominated by opening with $\beta_i (s_i^{\max} | \hat{b}^{i-1})$. On the other hand, opening above $\beta_i (s | \hat{b}^{i-1})$ presents no gain. Because $\beta_i (s | \hat{b}^{i-1})$ is continuous with full support on $[s_i^{\max}, \bar{s}]$, without loss of generality we can assume $\hat{b}_i = \beta_i (s | \hat{b}^{i-1})$ for some $s' \in [s_i^{\max}, \bar{s}]$. Let $\pi (s', y, s_i)$ denote $i$’s expected profit when $i$ opens with $\beta_i (s' | \hat{b}^{i-1})$ and exits at $y \geq \beta_i (s' | \hat{b}^{i-1})$, then $\pi (s', y, s_i)$ is given by equation 1.19. Let $i$’s best response be $(s^*, y^*)$. We then have

$$
(s^*, y^*) = \arg \max_{s' \geq s_i^{\max}, y \geq \beta_i (s' | \hat{b}^{i-1})} \pi (s', y, s_i).
$$

We solve for $(s^*, y^*)$ by maximizing $\pi (s', y, s_i)$ over a broader range $s' \geq s_i^{\max}$ and $y \geq s_i^{\max}$ and verify that the constraint $y \geq \beta_i (s' | \hat{b}^{i-1})$ is satisfied at the maximum point. Taking derivative with respect to $y$, we have $\frac{\partial \pi}{\partial y} = v (s', s_i) - y$. Now define $y^* (s') \equiv v (s', s_i)$. Then we have $\frac{\partial \pi}{\partial y} |_{y=y^* (s')} = 0$, $\frac{\partial \pi}{\partial y} |_{y>y^* (s')} < 0$ and $\frac{\partial \pi}{\partial y} |_{s_i^{\max} < y < y^* (s')} > 0$. Therefore, $y = y^* (s')$ maximizes $\pi (s', y, s)$ over $y \geq s_i^{\max}$ for any given $s$ and $s'$. Substituting $y = y^* (s')$ into $\pi (s', y, s_i)$, we have

$$
\pi (s', y^* (s'), s_i) = G_i (v (s', s_i)) v (s', s_i) - G_i \left( \beta_i (s' | \hat{b}^{i-1}) \right) \beta_i (s' | \hat{b}^{i-1}) - \int_{\beta_i (s' | \hat{b}^{i-1})}^{v (s', s_i)} Y_1 dG_i (Y_1).
$$

Taking derivative with respect to $s'$, we get
\[
\frac{\partial \pi(s', y^*(s'), s_i)}{\partial s'} = \lambda_i G_i (v(s', s_i)) - G_i \left( \beta_i (s'|\tilde b^{i-1}) \right) \frac{d\beta_i (s'|\tilde b^{i-1})}{ds'} = \lambda_i G_i (v(s', s_i)) - \lambda_i G_i (s'),
\]

where the second line follows from relation \( G_i \left( \beta_i (s'|\tilde b^{i-1}) \right) \frac{d\beta_i (s'|\tilde b^{i-1})}{ds'} = \lambda_i G_i (s') \), which results from differentiating equation 1.7. Note that \( v(s', s_i) > (\text{or} =, \text{or} <) s' \) if \( s' < (\text{or} =, \text{or} >) s_i \). Therefore, we have

\[
\frac{\partial \pi(s', y^*(s'), s_i)}{\partial s'} \bigg|_{s'<(\text{or} =, \text{or} >) s_i} > (\text{or} =, \text{or} <) 0.
\]

Therefore, \( s = s_i \) maximizes \( \pi(s', y^*(s'), s_i) \). Note that \( y^*(s_i) = s_i \). We then have

\[
(s^* = s_i, y^* = s_i) = \arg \max_{s_i \geq s_i^{\text{max}}, y \geq s_i^{\text{max}}} \pi(s', y, s_i).
\]

It is easy to verify that \( y^* \geq \beta_i (s'|\tilde b^{i-1}) \) is satisfied. Therefore, \( \beta_i (s'|\tilde b^{i-1}) \) and \( \gamma_i (s_i|\tilde b^{i-1}) \) as prescribed are the best response.

We now look at the case when \( s_i < s_i^{\text{max}} \). Note that the above analysis implies that the payoff from playing best response is zero if \( s_i = s_i^{\text{max}} \); therefore, the best response for \( s_i < s_i^{\text{max}} \) can at best generate zero profit. Since the strategies prescribed above result in a zero profit, they are the best response.

**Proof of Proposition 3**

Claim 1: The bidder with the highest synergy among all active bidders will participate in equilibrium.

Suppose bidder \( k \) has the highest synergy \( s_k \) among all active bidders. Let \( m \) denote the last participating bidder before \( k \). Note that \( k \) will participate as long as \( s_k \geq \max \{ b_k^{k-1} \} = \beta_m (s_m) \). Since \( \beta_m (s_m) \leq s_m \) and \( s_k \geq s_m \), the participating condition is satisfied and claim 1 is established.

In light of claim 1 and the fact that all participating bidders will exit at the truthful synergy, efficiency follows.

**Proof of Lemma 5**

The proof is given in the text.

**Proof of Proposition 4**

Denote by \( N \) the number of participating bidders. Let \( f_1 (\cdot) \) denote the marginal unconditional distribution of \( \tilde b_1 \), and let \( h (\cdot| \text{sin} ) \) denote \( \tilde b_1 \)’s distribution conditional on being single-bidder contest. Then from Bayes’ rule we have
\[ h(b_1|\sin) = f_1(b_1) \frac{\text{prob}(\sin|b_1)}{\text{prob}(\sin)}. \]

Note \( \text{prob}(\sin|b_1) \) increases in \( b_1 \). Therefore, the distribution \( h(\cdot|\sin) \) dominates \( f_1(\cdot) \) in likelihood, which implies \( E(b_1|\sin) > E(b_1) \). A similar argument shows \( E(b_1|\text{mul}) < E(b_1) \). Thus we have the proposition.

**Proof of Proposition 5**

We first prove the second part of the proposition. Since the participation threshold for bidder 1 is \( s \) in any equilibrium, assumption 5 implies that the participating threshold for any other bidder is also \( s \) in a pure pooling equilibrium. Therefore, all bidders will participate and the opening bid is independent of the synergy. This gives equation (i). Equation (ii) follows readily from proposition 1. Since \( \beta_i(s_i|b_i^{-1}) \leq \gamma_i(s_i|b_i^{-1}) \) for all \( s_i \), particularly for \( s_i = s \), this gives equation (iii). The market’s belief on the equilibrium path follows from equation 1.4.

We now prove the first part of the proposition by showing the following constitutes a pure pooling equilibrium:

\[
\begin{align*}
\gamma_i(s_i|b_i^{-1}) &= (1 - \lambda_i) s_i + \lambda_i E[s|s \geq s_i] \\
\beta_i(s_i|b_i^{-1}) &= \begin{cases} 
-1 & \text{if } \gamma_i(s_i|b_i^{-1}) < \max \{b_i^{-1}\} \\
\max \{b_i^{-1}\} & \text{otherwise.}
\end{cases}
\end{align*}
\]

The market’s belief is

\[ \theta_i = E[s|\gamma_i(s|b_i^{-1}) \geq p]. \]

The proof is straightforward. In equilibrium all bidders will participate by placing an opening bid of \( s \).

**Proof of Proposition 6**

First, from Lemma 3 pooling equilibria are efficient if \( \lambda_i \) is the same for all \( i \).

We now show that the equilibria are not efficient if \( \lambda_i \) is not the same for all \( i \). Suppose \( \lambda_i > \lambda_j \) without loss of generality. Then from proposition 1 the exit price for bidder \( k = i,j \) is

\[ \gamma_k = (1 - \lambda_k) s_k + \lambda_k E[s|s_k \leq s]. \]

It is thus possible to have \( \gamma_i > \gamma_j \) although \( s_i < s_j \), leading to inefficiency.

This establishes the proposition.

**Proofs of Lemmas 6 and 7**

The proofs are given in the text.

**Proof of Lemma 8**
If $s_i$ is independently revealed at $t = 1$, bidder $i$’s total proceeds from selling at $t = 1$ and $t = 2$ is just $s_i$. Since $p > s_i$, the proposition follows.

**Proof of Proposition 7**

The proposition follows from lemmas 6 through 8.

**Proof of Proposition 8**

Plugging $i = 1$ into equation 1.20 yields: $\lambda_i G_1(s) \, ds = G_1(\beta_1(s)) \, d\beta_1(s)$, and the boundary condition is $\beta_1(\underline{s}) = \underline{s}$. Since the expected number of bidders is more than 1, $G_1(s)$ strictly increases in $s$. Since $\beta_1(s) < s$ for all $s > \underline{s}$, $\beta_i(s|y^{i-1}) < s$ for all $s > \underline{s}$, we have $\frac{d\beta_1(s)}{ds} > \lambda_1$ for $s > \underline{s}$. Upon integration we get $\beta_1(s) > \underline{s} + \lambda_1 (s - \underline{s})$ for $s > \underline{s}$. Taking expectations on both sides, we have $\text{E}[\beta_1(s)] > \gamma^\text{min}$, which is the proposition.

**Proof of Lemma 9**

For any bidder $i$ with synergy $s_i = \underline{s}$, in cases $(i)$, the winning probability for the bidder is zero because of competition, and thus his expected profit vanishes.

We now discuss case $(ii)$. Since the equilibrium is pure separating, the bidder’s exit price is $\underline{s}$. If he wins, the price must be $p = \underline{s}$. In addition, his synergy $\underline{s}$ is fully revealed to the market. Therefore, his profit vanishes.

**Proof of Proposition 9**

Let $\Phi_i(s', s)$ be bidder $i$’s expected profit when he has synergy $s$ but follows the equilibrium strategy of type $s'$. Notice that $\Phi_i(s', s') - \Phi_i(s', s) = (1 - \lambda_i) G_{-i}(s')(s' - s)$, which implies $\frac{\partial \Phi_i(s', s)}{\partial s}|_{s = s'} = (1 - \lambda_i) G_{-i}(s)$; in addition, we have $\frac{\partial \Phi_i(s', s)}{\partial s'}|_{s' = s} = 0$ based on the FOC. Therefore, we have

$$\frac{d}{ds}|_{s = s'} \Phi_i(s', s) = (1 - \lambda_i) G_i(s).$$

Lemma 9 yields the boundary condition $\Phi(\underline{s}, \underline{s}) = 0$. We then have $\Phi(s, s) = (1 - \lambda_i) \int_{\underline{s}}^{s} G_{-i}(y) \, dy$. Since $\pi_i = q_i \int_{\underline{s}}^{s} \Phi(s, s) \, dF(s)$, we readily have the expression for $\pi_i$ as in the proposition. For the seller’s part, note $G_0(\cdot)$ is the distribution of the highest synergy among all bidders, and thus the first term in $\pi_s$ is the total welfare increase; the second term in $\pi_s$ is simply the expected sum of all bidders’ rents. We therefore have the claim.

**Proofs of Corollaries 10, 11, 12, 13, and 14**

They follow readily from the text.

**Proof of Proposition 10**

Any pure separating equilibrium is efficient from proposition 3.

We now show that any other equilibrium is not efficient. Note that if an equilibrium is not pure separating, a partition $(s_a, s_b)$ must exist on which the opening strategy is pooling. Suppose
bidder $i$ and $j$ have synergies $s_i, s_j \in (s_a, s_b)$. Then from proposition 1 the exit price for bidder $k = i,j$ is $\gamma_k = (1 - \lambda_k) s_k + \lambda_i \mathbb{E}[s | s_k < s < s_b]$. Suppose $\lambda_i > \lambda_j$ without loss of generality. It is thus possible to have $\gamma_i > \gamma_j$ although $s_i < s_j$.

The above establishes the proposition.

**Proof of Proposition 11**

If an equilibrium is not pure separating, a partition $(s_a, s_b)$ must exist on which the opening strategy is pooling. If $s_1 \in (s_a, s_b)$, bidder 1’s exit strategy is then $\gamma_1 (s_1) = (1 - \lambda_1) s_1 + \lambda_1 \mathbb{E}[s | s_1 < s < s_b]$. We have $\gamma_1 (s_1) > s_1$ for all $s_a \leq s < s_b$, leading to a positive probability of overpayment. This establishes the proposition.
1.9 Appendix B: Equilibria without Continuity Assumption

In this section I remove assumption 7 on continuity and discuss the main features of the general solution. Since \( \beta_1(\cdot) \) is nondecreasing, the set \([\underline{s}, \bar{s}]\) can be partitioned into intervals such that on each interval \( \beta_1(\cdot) \) is either separating or pooling. By assumption 5 this partitioning applies to all bidders’ opening strategies and thus can be used to classify the equilibria. Without loss of generality, one can always assume no two neighboring partitions are separating, because otherwise it would be possible to combine them into a single separating partition. The pure separating and pure pooling equilibria are the simplest equilibria with only one partition, and all hybrid equilibria have two or more partitions. To gain intuition on the hybrid equilibrium, consider an example of a simplest hybrid equilibrium with one bidder and two partitions, and that the opening strategy is separating in the first half of the synergy range and is pooling in the second half.

**Example 5.** Suppose \( n = 1, \lambda_1 = 0.5, \) and \( F \) is uniform on \([0, 1]\). Then the following forms a hybrid equilibrium:

\[
\beta_1(s_1) = \begin{cases} 
0.5s_1 & \text{if } 0 \leq s_1 \leq 0.5 \\
0.375 & \text{if } s_1 > 0.5 
\end{cases} \tag{1.24}
\]

and

\[
\theta_1(b_1, p) = \begin{cases} 
2b_1 & \text{if } 0 \leq b_1 \leq 0.25 \\
0.5 & \text{if } 0.25 < b_1 < 0.375 \\
0.75 & \text{if } b_1 \geq 0.375.
\end{cases} \tag{1.25}
\]

It is easy to verify that the above is an equilibrium. In this example \( s_1 \) is fully revealed if it is below 0.5; otherwise it is not. Therefore, the information release is intermediate. Note that \( \beta_1(s_1) \) is discontinuous at \( s_1 = 0.5 \), the boundary of the two partitions, because at the boundary, the bidder is indifferent between following separating or pooling strategy. Specifically, if he follows the separating strategy and opens with \( b_1 = 0.5 \times 0.5 = 0.25 \), his profit is \( 0.5 \times 0.5 + 0.5 \times 0.5 = 0.25 \); on the other hand, if he opens with \( b_1 = 0.375 \) instead, his profit is the same \( 0.5 \times 0.75 + 0.5 \times 0.5 = 0.25 \). In general, since the market’s belief is different for the two opening patterns, the discontinuity in the opening bid is necessary for the bidder to be indifferent.

In addition, a hybrid equilibrium necessarily contains a pooling partition because otherwise the equilibrium would be pure separating. Recall that in the case of pure pooling equilibrium,
the allocation is inefficient if not all $\lambda_i$ are the same, and overpayment is possible. These results generalize to any hybrid equilibrium.

**Proposition 10.** If $\lambda_i \neq \lambda_j$ for some $i$ and $j$, the pure separating equilibria are the only efficient equilibria satisfying assumptions 1 through 6.

**Proposition 11.** The pure separating equilibria are the only equilibria satisfying assumptions 1 through 6 that have no possibility of overpayment.
Table 1.2: List of frequently used notations.

1.10 Appendix C: List of Notations

Table 1.2 lists frequently used notations for reference.
Chapter 2

Hedging and Competition in a Project Market

This chapter is a joint work with Christine Parlour.

2.1 Introduction

Currently, NATO is tendering contracts to install an Active Layered Theatre Ballistic Missile Defense System. Different parts of this 680 million euro initiative are being awarded competitively for deployment in 2010. As this type of massive infrastructure investment becomes more common, firms actively compete in international markets for the rights to undertake such projects. These opportunities raise a number of theoretical and practical questions. Consider a US firm that plans to bid for a part of the ballistic defense system. First, the scale of such projects is typically large relative to existing investments and financing frictions are more relevant. Second, contracts are denominataed in euros and therefore winning exposes the US firm to exchange rate risk. Third, and most importantly, firms cannot control when or if they undertake the project; the decision will be made by NATO. In light of these three characteristics how should firms value the rights to undertake these projects? If a firm is planning to bid, what is the optimal hedging strategy? What implications do these types of investments have for industry outcomes? To analyze these questions we develop a model of the relationship between competition, investment, hedging and systematic risk in an industry with indivisible projects.

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While information on specific international investments is frequently confidential, aggregate overseas investment in 2004 was $222.4 billion dollars, according to U.S. Department of Commerce data at http://www.bea.gov/scb/pdf/2006/07July/0706_DIP_WEB.pdf.
We model firms in an industry competing for the rights to undertake a project. The value of the project has a private value component (a company specific synergy) and a common value component that is correlated with a common factor (such as, for example, exchange rate risk). The common value component is random and realized only if the project is undertaken; therefore winning generates uncertain internal capital. All firms in the industry face costly external financing and, in the normal course of business, invest in a concave production function. Therefore, each has an incentive to hedge cash flow risk to preserve internal capital for ongoing operations. Because the random common value component of the project is correlated with existing financial instruments, these can be used to lay off risk. However, hedging with financial instruments only eliminates cash flow risk if a firm wins the auction. Indeed, while all firms may find it optimal to hedge before the auction, all losers in the auction find themselves exposed to the common risk factor after the auction. This is because, in financial markets, contracts are not written contingent on who wins, therefore markets are incomplete with respect to who finally undertakes the project.

In the practitioner literature the risk faced by firms that hedge before they know if they have won an auction is known as the “bid to award period” risk. This timing friction arises from two sources. First, bids for complicated projects can take a long time to evaluate because factors other than price such as political considerations, a bidder’s technical credentials, and the payment schedule are relevant. For large and complex projects, a two-stage bidding process may even be employed where bidders are first invited to submit technical offers without prices, which are then evaluated to set an acceptable technical standard for all bidders, and then bidders are asked to resubmit bids with prices. Second, the bid to award period also includes the time in which the auction is anticipated but has not yet taken place: For example, many of the biggest procurements follow expired contracts. Recent surveys by Yee (2000) and Espinosa (2005) find the median bid to award period connected with FX risk to be between eight and nine months. Lidbark (2003) estimates that this time period generates a substantial risk exposure.

We provide closed-form solutions for optimal bids and hedging strategies in the presence of this friction. We predict that all firms should form the same hedging portfolio, but buy a dollar amount proportional to their chance of winning the project. We show that the existence of financial instruments in the presence of the timing friction makes firms compete more aggressively because by entering into a hedging position, bidders run the risk of being over hedged if they lose. We show that this effect can be so severe that a bidder may bid more than his actual

\footnote{An excellent description of this appears in Routledge(2003).}

\footnote{More details of one such evaluation process can be found in "Procurement Guidelines," Asian Development Bank, April 2006.}
synergy value for the project. In this framework, a bid translates directly into the amount that the winner invests into the project.

In addition, we show that the ability to hedge makes industry outcomes more, rather than less, variable. For example, hedging increases the variance of the bids. Bidders with larger private values have a higher probability of winning and therefore hedge more ex-ante. The more a firm has hedged the more costly losing becomes and therefore bids are increasingly more aggressive. This effect also increases the dispersion in ex post firm values. Finally, we show that the spillover between projects can affect firms’ investment in projects that they do not compete over. With hedging, the covariance of internal capital changes with the risk factor is negative, and is more negative, the higher the correlation of the hedging instrument with the risk factor.

We also examine bidders’ and the seller’s profits. We show that any benefits to bidders from hedging instruments accrue to the seller (they compete them all away). We also show the timing friction decreases the seller’s profit: the winner is underhedged and the losers are overhedged, and the loss in social welfare is born by the seller. This has both normative and positive implications. Normatively, it is to the seller’s advantage to accelerate the evaluation process (to reduce the “bid to award” period) or to hedge the common value of the project himself. Positively, this suggests that the framework is most appropriate in markets in which the seller also faces financing constraints and therefore does not provide the hedge or finds it optimal for bidders to retain some risk. Such contingent hedges could never be provided by a competitive market. If contracts are contingent on which firms win an auction then agents’ payoffs are conditional on actions they take. That is, an agent can control if he wins the auction through his bid: as in the case of competitive insurance markets moral hazard would lead to market breakdown.

A large literature studies corporate risk management. Stulz (1984), Smith and Stulz (1985), Stulz (1990), Bessembinder (1991), and Froot et al. (1993) find that hedging increases firm value in the presence of market imperfections such as costs of external financing and a progressive tax rate schedule. Given these frictions, the optimal hedging strategies are analyzed. Such studies of optimal hedging strategies posit that the firm’s cash flows are a deterministic function of some common factors and the fluctuations in cash flow come solely from the fluctuations of the factors. In our framework, a firm’s cash flow is uncertain conditional on the common factor and it depends on whether or not the firm has won the auction.

The relationship between hedging and product market competition has been analyzed by Adam, Dasgupta and Titman (2005). They characterize Cournot equilibrium in which firms may hedge input costs. They find that hedging strategy is jointly determined in equilibrium with the product market competition and that there can be asymmetric equilibria. Mello and
Ruckes (2005) also consider firms with financial constraints who compete in product markets. They find that even though reducing cash flow volatility may be desirable, firms may choose not to hedge for competitive reasons. We find the reverse, that firms in active competition will always hedge and ex post those who lose the competition have too much exposure to the common factor. Loss (2002) demonstrates that a firm in competition chooses to hedge strategically, in particular, hedging is valuable if investments are strategic substitutes (as opposed to strategic complements). Conceptually, we differ from these papers in one important way. We consider competition in indivisible projects. This means that only one firm will “win,” and the other firms will necessarily “lose.”

A literature has considered the interaction between hedging and bidding. Eaker and Grant (1985) model a firm with an exogenous probability of winning a project whose payoff is exposed to foreign exchange rate risk. They study the optimal exposure to a forward contract on the exchange rate. In a similar framework, Lien and Wong (2004) incorporate a single bidder facing an exogenous function that links bids to winning probabilities. We extend this literature, by rendering the bids and hedging endogenous. Therefore, we can consider the interaction between financial markets on firms’ bidding behavior.

This paper also contributes to the auction literature through the observation that, through financial markets, agents may affect their valuations. The incentive to hedge in this model arises because firms face a convex cost of capital. Thus, the model resembles auction models with risk averse bidders. Eso and White (2004) examine the effect of idiosyncratic risk on risk averse bidders. They find that idiosyncratic risk can increase agents’ expected utility.

Rhodes-Kropf and Viswanathan (2005) also consider bidders who are financially constrained. In their framework, bidders with differing cash positions raise money which may be conditional on the ex post value of the firm. They show that capital markets may render the auction inefficient. By contrast to their work, as we are interested in how competition affects the investment decisions of firms, we assume that the cost of accessing the financial market either through hedging or raising extra capital does not depend on the bid or private information of the bidders.

The rest of the paper is organized as follows. We present the model in Section 2.2, and in Section 2.3, we explicitly solve the model and derive testable predictions in Section 2.4. We discuss the welfare implication and seller revenue in Section 2.5 and finally we conclude in Section 2.6. Most proofs are in the first appendix, Section 2.7. A general existence proof for the case of a concave growth opportunity appears in Section 2.8. Some comments on the robustness of our modeling assumptions follow in the third appendix, Section 2.9.
2.2 The Model

Consider a model with three dates, \( t = 0, 1, 2 \), no discounting, and universal risk neutrality. \( N \) identical firms, indexed by \( j = 1, \ldots, N \), with initial internal funds \( W_0 \) compete in an industry made up of two types of investment opportunities. One is indivisible and awarded competitively by an auction. We refer to this as “the project.” The other is on-going and available to all firms at time \( t = 1 \). We describe this as a “growth opportunity.” Thus our model captures firms in competition in a well-established industry.

At \( t = 0 \), each firm observes its private value for the project, \( s_j \). This value is independently drawn across firms from distribution \( F \) on \([s, \bar{s}]\) and represents firm specific economies of scale (or synergies) in undertaking the project. Let \( G(s) = F^{N-1}(s) \) denote the distribution of the highest signal among \( N - 1 \) firms.

The project is awarded in a second price auction. Let \( \beta(s_j) \) denote the bidding strategy of a firm with signal \( s_j \). If a firm wins the auction, it receives its private value at \( t = 2 \) at which time it pays for the project. We denote the random payment amount by \( \tilde{b} \), which is the highest losing bid.

At \( t = 0 \), after observing their signals, firms privately trade in financial contracts at the time they bid. The \( i \in \{1, \ldots, n\} \) financial contracts have \( t = 1 \) payoffs contingent on \( \tilde{x} \), denoted \( h_i(\tilde{x}) \). We assume that \( \tilde{x} \) is a random factor that is correlated with the project’s common value, \( \omega \). Thus, \( \tilde{x} \) could represent foreign exchange. Here, \( \omega \) is a random variable drawn from \([\underline{\omega}, \bar{\omega}]\) with \( E[\omega] = 0 \). This value accrues to the auction’s winner at time \( t = 1 \) and it can be interpreted as an “announcement effect”: the part of the project’s value that is independent of the firm which undertakes it. Without loss of generality, we assume that the price of each financial contract is zero, or equivalently, that the interest rate is zero.

Let \( q_i \) denote the quantity of contract \( i \) purchased by a firm. A firm’s hedging strategy is denoted by \( \{q_i(s_j)\}_{i=1}^n \).

Assumption 8. (i) The contracts are unbiased so that \( E[h_i(\tilde{x})] = 0 \), for all \( i = 1, \ldots, n \).
(ii) The contracts are non-cancelable: If (i) is satisfied and \( Pr(\Sigma_{i=1}^n q_i h_i(\tilde{x}) = 0) = 1 \), then \( q_i = 0 \) for all \( i \).

The first part of the assumption is that there are no transaction costs associated with the hedging instruments. That is, the expected payoff to each contract is equal to the cost. We use \( Q \) to denote the set of \( \{q_i\}_{i=1}^n \) satisfying assumption 8 part (i). The second part precludes degenerate solutions in which firms buy arbitrary and offsetting amounts of unbiased hedging instruments, in which case the solution is indeterminate.
The hedging instruments pay out at $t = 1$. At the same time, the auction winner is announced and $\tilde{\omega}$ is realized. For a firm with exposure $q_i$ to $n$ hedging instruments, its time $t = 1$ internal capital is

$$W_1 = \begin{cases} W_0 + \tilde{\omega} + \sum_{i=1}^n q_i h_i(\tilde{x}) & \text{wins} \\ W_0 + \sum_{i=1}^n q_i h_i(\tilde{x}) & \text{loses.} \end{cases} \quad (2.1)$$

This internal capital is available for investment in the growth opportunity.

A firm’s valuation of the $t = 1$ growth opportunity determines how it competes for the project at $t = 0$. All firms face the same growth opportunity. To generate a hedging motive we adapt the framework of Froot, Scharfstein and Stein (1993) which show how a convex cost of external capital can be represented in reduced form by a concave function of internal capital. Let $I(W_1)$ denote a firm’s valuation of the growth opportunity, where $W_1$ is its internal capital at $t = 1$. To provide precise results, we assume a firm’s growth opportunity is quadratic of the form $I(W_1) = a + bW_1 - cW_1^2$ where $c > 0$. We denote the marginal value of internal capital at $W_0$ by $m = I'(W_0) = b - 2cW_0$, which we assume is always positive. Empirically, Altinkilic and Hansen (2000) find that the cost of external funds is consistent with the quadratic form, and the quadratic form has also been used in Hennessy and Whited (2006). Theoretically, for any general function, $I(\cdot)$, if the curvature is small then a Taylor series expansion up to the quadratic term is a good approximation.

Given our formulation, the time $t = 2$ value of a firm with signal $s_j$, and exposure $q_i$ to $n$ hedging instruments is

$$V(s_j) = \begin{cases} I(W_0 + \tilde{\omega} + \sum_{i=1}^n q_i h_i(\tilde{x})) + s_j - \bar{b} & \text{firm } j \text{ wins} \\ I(W_0 + \sum_{i=1}^n q_i h_i(\tilde{x})) & \text{firm } j \text{ loses} \end{cases} \quad (2.2)$$

As the financial contracts are not written conditional on who wins the auction, the loser (if hedged) is exposed to $\tilde{\omega}$ risk. This increases the variability of a firm’s internal funds relative to the case in which the firm does not participate in the auction. As external funds are costly (captured by the concavity of the payoff to the growth opportunity) firms are worse off in expectation if they lose the auction.

The sequence of events is depicted in Figure 2.1.

The time between $t = 0$ and $t = 1$, is the timing friction called “the bid to award period.” The longer it is, the more bidding firms are exposed to fluctuations in $\tilde{\omega}$.\footnote{For example, if $\tilde{\omega}$ evolves according to a Brownian motion then the $t = 1$ variance of $\tilde{\omega}$ will be proportional to the length of the bid to award period.}

We have assumed that firms pay for the project and receive their private benefit in period
Figure 2.1: Sequence of events

In other words, we assume that the firm’s private synergy value of the project, \(s_j\) and the cost of the project, \(\tilde{b}\), do not affect time \(t = 1\) internal capital. While it is quite natural to assume that private synergies are realized over time therefore do not immediately affect internal capital,\(^5\) the assumption that payment occurs later is stronger. We primarily make this assumption for tractability. An alternate timing assumption would be that firms pay their bid and receive the private benefit at \(t = 1\). This timing assumption complicates the analysis because firms behave as if risk averse at \(t = 0\) (due to the concavity of the growth opportunity, \(I(W_1)\)). In a second price auction, the payment is a random variable. In a first price auction, payments are not random but there are few existence results for auctions with concave payoff functions (for example, see Reny and Zamir (2003) and McAdams (2007)), and the situation is even more complicated with hedging instruments. However, we find that our results are qualitatively robust to alternate timing assumptions. In Section 2.9 we present robustness results with specific functional forms and alternate timing assumptions.

### 2.3 Characterization of Equilibrium

#### 2.3.1 Benchmark Bids

For now, we abstract from the hedging positions that firms take and consider the bids. In the standard second price auction, the bidder is indifferent between losing and winning at the price of the bid. The same intuition applies with hedging: given the firm’s hedging strategy, the bid is the difference in the expected value of the growth opportunity if the firm wins the auction and the expected value if the firm loses the auction.

**Lemma 15.** Consider a bidder with signal \(s_j\) and an exogenous hedging amount \(\{q_i\}_{i=1}^n\), then

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\(^5\)If the private signals are unverifiable and uncontractible, the firm must wait for them to be realized before its internal capital is affected. By contrast, as the \(\tilde{\omega}\) component is the same for all firms, this affects its current cashflows.
bidding
\[ \beta(s_j \mid \{q_i\}_{i=1}^n) = s_j + E[I(W_0 + \tilde{\omega} + \Sigma q_i h_i(\tilde{x}))] - E[I(W_0 + \Sigma q_i h_i(\tilde{x}))] \] 

(2.3)
is a (weakly) dominant strategy.

Hedging increases the bid in two ways. It increases the value of the project conditional on winning while at the same time decreases the value upon losing.

Consider an economy in which there are no financial contracts available and therefore no hedging (NH). The bid function (by inspection of lemma 15) is just

\[ \beta^{NH}(s_j) = s_j + (E[I(W_0 + \tilde{\omega}) - I(W_0)]) \]

Given that \( \tilde{\omega} \) has mean zero, and that the investment function is concave, the bid function is less than \( s_j \).

Another natural benchmark economy is one in which financial contracts can be written contingent on winning the auction. The hedging strategy in this economy is straightforward: all bidders hedge as if they would win with certainty. Since only the winner actually receives payoffs from the hedging instruments, effectively only the winner hedges in this economy. In this case the bid function with contingent hedging, \( \beta^C(s_j) \), is less than the bidder’s actual signal \( s_j \). However, it is greater than the bidding function with no hedging. This is because the opportunity to hedge risk increases the value of the project to the firm. Thus,

\[ \beta^{NH}(s_j) \leq \beta^C(s_j) \leq s_j. \]

If there is hedging, but it is not contingent, then bids are also higher than in the no hedging case. There are two reinforcing effects: firms’ valuation for the project are higher if they can hedge the risk. In addition, a hedged firm has more to lose if it does not win the auction because losing exposes it to the factor \( \tilde{x} \). As the price that a firm pays for the project is the highest losing bid, this implies that the presence of financial contracts leads firms to pay more (i.e., invest more) in the project.

**Proposition 12.** The equilibrium bid of any firm in an economy with hedging vehicles available is higher than an equivalent firm in an economy with no hedging instruments available. Or, \( \beta(s_j) \geq \beta^{NH}(s_j) \), for all \( s_j \).
2.3.2 Optimal Hedging and Bidding

In order to maximize expected profits, a firm hedges so as to optimally dampen variation in internal capital. Consider a firm with a fixed probability, \( p \), of winning the auction. If \( p = 0 \), the firm does not purchase any financial contracts because it will lose and if it hedges it exposes itself to systematic \( \tilde{x} \) risk. By contrast, if \( p = 1 \) the firm wins for sure and optimally hedges to minimize the exposure to \( \tilde{x} \). However, if \( p \in (0,1) \), the firm faces a tradeoff. Thus, the optimal hedging amounts \( \{ q^*_i \}_{i=1}^n \in Q \) are the solution to:

\[
\{ q^*_i (p) \}_{i=1}^n = \arg \max_{\{ q_i \}_{i=1}^n \in Q} \{ pE [I(W_0 + \tilde{\omega} + \Sigma q_i h_i(\tilde{x}))] + (1-p)E [I(W_0 + \Sigma q_i h_i(\tilde{x}))] \}.
\]

Firms hedge to tradeoff the payoff in states in which they win the auction and the ones in which they lose.\(^6\) A bidder with a higher probability of winning optimally changes hedging amounts so that the expected value of the growth opportunity conditional on winning increases, while that conditional on losing decreases.

**Lemma 16.** Let \( \{ q'_i \}_{i=1}^n \) be the optimal hedging amounts for \( p' \), and let \( \{ \tilde{q}_i \}_{i=1}^n \) be the optimal hedging amount for \( \tilde{p} \). If \( \tilde{p} > p' \), then

(i) \( E [I(W_0 + \tilde{\omega} + \Sigma \tilde{q}_i h_i(\tilde{x}))] \geq E [I(W_0 + \tilde{\omega} + \Sigma q'_i h_i(\tilde{x}))] \)

(ii) \( E [I(W_0 + \Sigma \tilde{q}_i h_i(\tilde{x}))] \leq E [I(W_0 + \Sigma q'_i h_i(\tilde{x}))] \).

If there are many hedging instruments available, then a firm forms a portfolio to achieve the optimal hedge. Let \( \theta(\tilde{x}) \) be the portfolio with the highest possible correlation with \( \tilde{\omega} \) out of all possible portfolios that has the same variance as \( \tilde{\omega} \). We dub this the maximum correlation portfolio.

**Definition 3.** The maximum correlation portfolio, \( \theta(\tilde{x}) = \Sigma_{i=1}^n z_i h_i(\tilde{x}) \), is the portfolio of financial instruments that is maximally correlated with \( \tilde{\omega} \) so that

(i) \( \text{corr}(\tilde{\omega}, \Sigma z_i h_i(\tilde{x})) \geq \text{corr}(\tilde{\omega}, \Sigma z'_i h_i(\tilde{x})) \) for \( \forall \{ z'_i \}_{i=1}^n \)

(ii) \( \text{var}(\Sigma_{i=1}^n z_i h_i(\tilde{x})) = \sigma_{\tilde{\omega}}^2 \).

Let \( \rho \equiv \frac{\sigma_{\tilde{\omega}, \theta(\tilde{x})}}{\sigma_{\tilde{\omega}} \sigma_{\theta(\tilde{x})}} \), the correlation coefficient between the maximum correlation portfolio and the risk factor \( \tilde{\omega} \). Different industries face different common factors in their project cash flows. An industry with a larger \( \rho \) has project cash flows that are more correlated with financial instruments. If there is no hedging vehicle available, then \( \rho = 0 \). An industry’s financial capacity also affects

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\(^6\)Mello and Parsons (2000) argue that hedging transfers wealth across states therefore increases firm’s financial flexibility.
\( \rho \). For example, given margins, a firm with high leverage may not have the ability to put on futures contracts, rendering \( \rho \) low.

We show that each firm acquires the maximum correlation portfolio, but the dollar value that each firm invests in it depends on the probability that it wins the project.\(^7\) Specifically, the position that a firm takes depends on the probability that it has the highest value conditional on the private signal, \( s_j \). Recall, \( G(\cdot) \) is the distribution of the highest signal among the other \( N - 1 \) bidders. Thus \( G(s_j) \) is the probability that a firm with signal \( s_j \) has the highest signal. As this auction is efficient, the firm with the highest signal wins the auction, therefore \( G(s_j) \) is the probability that \( s_j \) wins. We adhere to the convention that firm 1 has the highest realized signal and the firm with \( s_1 \), wins the auction.

**Proposition 13.** If the \( n \) hedging instruments are unbiased and \( I(\cdot) \) is quadratic, then a firm with signal \( s_j \)

1. hedges \( q(s_j) = -G(s_j) \rho \) of portfolio \( \theta(\bar{x}) \)
2. and bids \( \beta(s_j) = s_j - c \sigma^2 \omega [1 - 2G(s_j)\rho^2] \)
3. whereas if hedging is contingent then a firm with signal \( s_j \)
4. hedges \(-\rho \) of portfolio \( \theta(\bar{x}) \) if he wins and 0 otherwise
5. and bids \( \beta^C(s_j) = s_j - c \sigma^2 \omega [1 - \rho^2] \).

Figure 2.2 plots bids as a function of the signals assuming that \( N = 2 \), \( c \sigma^2 \omega = 0.4 \), \( \rho^2 = 0.7 \) and that signal is uniformly distributed on \([0, 1]\). The solid line is the optimal bid. The bottom dashed line plots the bids in an economy without hedging instruments (NH), the middle dashed line plots the bids in an economy with contingent hedging (C) and the top dashed line plots the bidder’s actual signal.

Notice, that in this example, \( \beta(s_j) > \beta^C(s_j) \) for \( s_j > \frac{1}{2} \). More generally, it follows from proposition 13 that \( \beta(s_j) > \beta^C(s_j) \) for \( G(s_j) > \frac{1}{2} \). In addition, in this example, a bidder with a signal large enough will bid even more than his actual value, or \( \beta(s_j) > s_j \). Once again, access to non-contingent hedging instruments makes the bids more aggressive, especially for bidders with large signals, because hedged bidders have more to lose.

### 2.4 Predictions

To quantify the industry wide effects of hedging, we define a sample variance for \( N \) realized variables as

\(^7\)This follows from our assumption that the random component of the cash flow is the same for all firms.
Figure 2.2: The solid line plots the bids as a function of the bidder’s signal assuming that $N = 2$, $c_\sigma^2 = 0.4$, $\rho^2 = 0.7$ and that signal is uniformly distributed on $[0, 1]$. The bottom dashed line plots the bids in an economy without hedging instruments, the middle dashed line plots the bids in an economy with contingent hedging and the top dashed line plots the bidder’s actual signal.

**Definition 4.** The sample variance for $\{x_i\}_{i=1}^N$ is

$$\Sigma_x \equiv \frac{1}{N - 1} \sum_{i=1}^N \left( x_i - \frac{1}{N} \sum_{j=1}^N x_j \right)^2.$$ 

If the sample is an i.i.d draw, the expected sample variance corresponds to the distribution variance. Under this definition, we show formally that the ability to hedge increases the expected sample variance for the bids. In other words, an econometrician estimating the variance of bids in an industry, on average would report a higher sample variance if firms have access to hedging instruments.

**Proposition 14.** If the $n$ hedging instruments are unbiased and $I(\cdot)$ is quadratic, then the ability to hedge increases the expected sample variance for the bids. Or, $E[\Sigma_\beta] > E[\Sigma_\beta^{NH}]$.

The industry wide dispersion in bids is larger if firms can hedge. This is because ex-ante a bidder with a larger signal has a higher probability of winning and hence will hedge more. As a result, he has more to lose and his bid will be increasingly more aggressive.
From Proposition 13 the optimal hedging strategy for a firm is to scale the hedge in an economy in which the hedging instruments are contingent by the probability that it will win the auction. Thus, each firm hedges with the same portfolio and the amount is proportional to its equilibrium probability of winning.\(^8\) Thus, the firm that wins the auction hedges the most, and among firms that lose, the amount each firm hedges differs. This leads to an observed dispersion of hedging positions in the industry.

**Proposition 15.** The expected sample variance of relative hedging positions in an industry is

\[
E[\Sigma_h] = \rho^2 \frac{(N-1)^2}{N^2(2N-1)}.
\]

On average, the dispersion in hedging positions depends on both the number of firms in the industry and the availability of highly correlated financial instruments. If \(\rho\) is high, then the dispersion is higher. Or, the easier it is to lay off risk, the more diverse the expected outcomes.

After the project has been awarded, ex ante identical firms hold different internal funds. Thus, each firm raises different amount of external capital and invests different amounts in the growth opportunity. Let \(\Delta W(s_j)\) denote the change in a firm’s internal capital, i.e., \(\Delta W(s_j) = W_1(s_j) - W_0\). We have

\[
\Delta W(s_j) = \begin{cases} 
\bar{w} - G(s_j)\rho\theta(\bar{x}) & \text{firm } j \text{ wins} \\
-G(s_j)\rho\theta(\bar{x}) & \text{firm } j \text{ loses.}
\end{cases}
\]

The internal funds of the auction’s winner are only partially hedged while all losers are over-hedged. Furthermore, among the losing firms, the one with a larger signal \(s\), is more overhedged. Therefore the winner’s internal capital will tend to be either the largest or smallest among all firms depending on the realization of \(\bar{\omega}\). For example, if \(\bar{\omega}\) is positive, the winner’s internal capital is likely to be above \(W_0\) while the losers’ is likely to be below \(W_0\). Hence, a plot of the investment in the growth opportunity against the private signal \(s\) for all the losing firms will be monotonic with the slope depending on the realization of \(\bar{\omega}\). If we proxy the firm’s private benefit by firm size or quality, then the losing firm’s investment varies monotonically with firm size or firm quality.

We can calculate the sample variance of changes in internal capital \(\Sigma_{\Delta W}\).

---

\(^8\)This is consistent with Eaker and Grant (1985) who consider optimal hedging in the quadratic case with exogenous probabilities of winning.
Figure 2.3: The expected sample variance of changes in internal capital as a function of $N$. The middle curve is the general case, and the top and bottom curves are for no hedging and contingent hedging respectively. Parameters $\sigma_\omega = 1$ and $\rho^2 = \frac{1}{2}$ are used.

**Proposition 16.** The expected sample variance of changes in internal capital is

$$\frac{\sigma^2}{N} \left( 1 - \rho^2 \frac{N^2-1}{(2N-1)N} \right),$$

where

(i) $\frac{\sigma^2}{N}$ is the sample variance if there are no hedging instruments available,

(ii) $\frac{\sigma^2}{N} (1 - \rho^2)$ is the expected sample variance if hedging is contingent.

Figure 2.3 plots the expected sample variance of changes in internal capital as a function of $N$ in the middle curve, and the top and bottom curves are for the benchmark cases of no hedging and contingent hedging respectively. Parameters $\sigma_\omega = 1$ and $\rho^2 = \frac{1}{2}$ are used.

The variance of internal cash flows is highest if there are no hedging instruments available, and lowest if the hedging instruments are contingent.

Even though only one firm (the winner) takes on the project, as all firms hedge, there is an industry–wide negative covariance between $\omega$ and changes in internal capital: hedging introduces a negative component in a regression of innovations in internal capital on systematic risk. Further, the size of the reduction depends on the correlation of the hedging instruments with the risk factor. Firms’ internal capital becomes more negatively correlated to the common factor if $\rho$ is higher.
Proposition 17. The covariance of innovations in internal capital with the risk factor is smaller, the more effective the hedging instruments. Or, for \( \rho > \rho' \),

\[
E[\Delta W(s_j \mid \rho)\omega] < E[\Delta W(s_j \mid \rho')\omega].
\]

2.5 Social Welfare

The two benchmark cases of no hedging and contingent hedging represent the lower and upper bounds on social welfare in the economy. Ex ante social efficiency is obtained when the firm with the highest synergy value wins the project and no firm has uncertain cash flows at \( t = 1 \).

While hedging makes bids more aggressive, it does not increase expected firm profits. Specifically, firms bid more aggressively because hedging increases (conditional on winning) the value of the growth opportunity. Thus, while the value of winning is higher, so is the cost as all firms adjust their bids accordingly. In equilibrium, these two effects are exactly offsetting. This arises because of the linearity of a bidder’s profit in the private synergy value and the bid payment.

Proposition 18. Consider an economy with a fixed number of hedging instruments. If an additional hedging instrument is added, then each firm’s expected profit remains unchanged at

\[
E[V(s_j)] = I(W_0) + \int G(z)dz.
\]

Therefore, a firm’s ex-ante expected value is independent of the availability of hedging instruments. However, this is not true for firms’ values ex-post. In particular, the ability to hedge increases the winner’s profit and decreases a loser’s. This is because hedging makes the losers of the auction much worse off, but the winner is much better off because ex-ante a bidder anticipates the expected loss conditional on losing, and therefore he demands a higher premium upon winning. Therefore hedging increases the dispersion in the ex-post firm values.

Proposition 19. In an \( N \) firm industry with quadratic investment opportunities, the winner’s expected value is larger than in an economy without hedging instruments, and any loser’s expected value is smaller. Specifically, firm value differences are given by

\[
E[V] - E[V^{NH}] = \begin{cases} 
\frac{N(N-1)}{(2N-1)(3N-2)}c^2\sigma^2\rho^2 & \text{for the winning firm} \\
-\frac{N}{(2N-1)(3N-2)}c^2\sigma^2\rho^2 & \text{for a losing firm.} 
\end{cases}
\]

Recall that if any new instruments are added to the economy, bidders’ expected profits do not increase. However, indifference to the number of hedging instruments is not true for the seller: a seller is better off if there are extra instruments available. This follows immediately from the
fact that social welfare must weakly increase if there are more hedging instruments, but bidders’
profits are unchanged. The expected revenue to the seller is the expected value of the second
highest bid:

\[ \pi_s = \int_{s_2}^{\hat{s}} \beta(s_2) dF_2(s_2), \]

where \( s_2 \) is the second highest order statistic and \( F_2(\cdot) \) is its distribution. Thus, if bidders bid
more aggressively, then the seller garners extra revenue. On the other hand, the seller’s expected
profit is still less than in the case of contingent hedging.

**Proposition 20.** If an additional hedging instrument is added, the seller’s expected revenue
(weakly) increases, but is less than with contingent hedging, or, \( \pi^{NH}_s \leq \pi_s \leq \pi^C_s \). The reduction
in seller’s profit because the hedging instruments are not contingent is

\[ \pi^C_s - \pi_s = c\sigma^2 \beta^2 \frac{N-1}{2N-1}. \]

We can put bounds on the seller’s profits. Specifically, the seller’s profit lies between \( \pi^{NH}_s \),
the no hedging profit and \( \pi^C_s \). Intuitively, this is because the social welfare lies between the two
poles, and any changes in the social welfare accrue entirely to the seller.

The seller’s profit loss because hedging instruments are not contingent has both normative
and positive implications. Normatively, it is to the seller’s advantage to accelerate the evaluation
process or to hedge the common value of the project himself, if the cost of raising capital is lower
for the seller than for the buyers. Positively, this suggests that the framework is most appropriate
in markets in which the seller also faces financing constraints and therefore does not provide the
hedge or finds it optimal for bidders to retain some risk. For example, if firms are competing to
acquire a going concern, the target is usually much smaller than any potential acquirers and more
financially constrained, especially if the target is already in financial distress. Also, in the case
of government tendering projects, the political costs of insuring bidders would be prohibitively
high. Finally, in procurement auctions, there may be moral hazard reasons that make it optimal
for a seller to ensure buyers retain some risk.

### 2.6 Conclusion

We have exhibited equilibrium bidding and hedging strategies in a second price auction with
random payoffs and a bid-to-award lag. Bidders know they will face a convex cost of capital
in the future, and have an incentive to hedge the cash flow as they bid. Because the bidders’
hedging decisions are made prior to the award announcement, they face a basic trade off between
maximally hedging the cash flow in the event of winning the auction and not hedging any amount
in the case of losing. As a result, ex-ante hedging is optimal for a firm, but ex-post the losers are exposed to a common risk factor.

The ability to hedge makes firms compete more aggressively because by entering into a hedging position, bidders run the risk of being over hedged if they lose. Indeed, this effect can be strong enough such that the bid may even exceed that which obtains when contracts can be written contingent on winning the auction. Furthermore, the ability to hedge may make certain outcomes more rather than less variable. For example, the variance of the bids is increased. This is because a bidder with a larger private value for the project has a larger winning probability and will hedge more ex-ante, and this results in larger losses for him upon losing and therefore his bid is increasingly more aggressive. In addition, hedging also increases the dispersion in the ex-post firm values because hedging makes the loser much worse off, and that in turn makes the winner much better off. Furthermore, the spillover between projects can affect firms’ investment in projects that they do not compete over. With hedging the covariance of internal capital changes with the risk factor is negative, and is more negative, the higher the correlation of the hedging instrument with the risk factor.

Because bidders have to make hedging decisions prior to knowing the auction outcome, social welfare is lower. This loss is born by the seller, and therefore reduces the seller’s profit. This result has both normative and positive implications. Normatively, it is to the seller’s advantage to accelerate the evaluation process or to hedge the common value of the project himself, if the cost of raising capital is lower for the seller than for the buyers. Positively, this suggests that the framework is most appropriate in markets in which the seller also faces financing constraints and therefore does not provide the hedge or finds it optimal for bidders to retain some risk.

Overall, this paper presents a framework in which the effect of a financing friction is magnified in an industry due to competition. All firms hedge as they anticipate future cash flow constraints. Due to competition, firms bid away benefits to hedging and in aggregate exacerbate the industry’s constraints.
2.7 Proofs

Proof of Lemma 15

Suppose that a bidder with signal $s$, and hedging instruments, $\{q_i\}_{i=1}^n$, bids $b$. Let $K(\cdot)$ denote the distribution of the highest bid among the remaining $N-1$ bidders. Then, his expected profit, $E[V(s_j)]$ is

$$\int_0^b [s - z + E[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))]] dK(z) + (1 - K(b)) E[I(W_0 + \Sigma q_i h_i(\bar{x}))].$$

Hence,

$$\frac{\partial E[V(b)]}{\partial b} = k(b) [s - b + E[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))] - E[I(W_0 + \Sigma q_i h_i(\bar{x}))]].$$

Equation 2.4 is (weakly) decreasing in $b$ and zero when the bid is as in the lemma.

Proof of Proposition 12

First, observe that the optimal holding of hedging instruments is zero if $s = \underline{s}$. Thus, for any $s > \underline{s}$, from lemma 16

$$E[I(W_0 + \bar{\omega} + \Sigma q_i(s) h_i(\bar{x}))] \geq E[I(W_0 + \bar{\omega})],$$

and

$$E[I(W_0 + \Sigma q_i(s) h_i(\bar{x}))] \leq I(W_0).$$

Therefore, equation 2.3 implies

$$\beta(s) \geq s + E[I(W_0 + \bar{\omega})] - E[I(W_0)] = \beta^{NH}(s).$$

Proof of Lemma 16

Recall that $\{q^*_i\}_{i=1}^n$ is the optimal hedging amounts if the probability of winning is $p'$, and $\{\widehat{q}_i\}_{i=1}^n$ is the optimal hedging amounts for $\hat{p}$ where $\hat{p} > p'$. By the definition of optimality;
\[
p' E[I(W_0 + \bar{\omega} + \Sigma q'_i h_i(\bar{x}))] + (1 - p') E[I(W_0 + \Sigma q'_i h_i(\bar{x}))] \geq p' E[I(W_0 + \bar{\omega} + \Sigma q h_i(\bar{x}))] + (1 - p') E[I(W_0 + \Sigma q h_i(\bar{x}))]
\]

and

\[
\tilde{p} E[I(W_0 + \bar{\omega} + \Sigma q h_i(\bar{x}))] + (1 - \tilde{p}) E[I(W_0 + \Sigma q h_i(\bar{x}))] \geq \tilde{p} E[I(W_0 + \bar{\omega} + \Sigma q'_i h_i(\bar{x}))] + (1 - \tilde{p}) E[I(W_0 + \Sigma q'_i h_i(\bar{x}))].
\]

Adding the two inequalities, and using the fact that \( \tilde{p} > p' \) yields

\[
E[I(W_0 + \bar{\omega} + \Sigma q h_i(\bar{x}))] + E[I(W_0 + \Sigma q h_i(\bar{x}))] \geq E[I(W_0 + \bar{\omega} + \Sigma q h_i(\bar{x}))] + E[I(W_0 + \Sigma q h_i(\bar{x}))].
\]

We prove part (i) of the lemma by contradiction. Suppose it is not true so that

\[
E[I(W_0 + \bar{\omega} + \Sigma q h_i(\bar{x}))] > E[I(W_0 + \bar{\omega} + \Sigma q'_i h_i(\bar{x}))].
\]

Then, Inequality 2.6 implies that

\[
E[I(W_0 + \Sigma q h_i(\bar{x}))] > E[U(W_0 + \Sigma q'_i h_i(\bar{x}))].
\]

Summing inequalities 2.8 and 2.9, yields

\[
E[I(W_0 + \Sigma q h_i(\bar{x}))] + E[I(W_0 + \bar{\omega} + \Sigma q h_i(\bar{x}))] > E[U(W_0 + \Sigma q'_i h_i(\bar{x}))] + E[I(W_0 + \bar{\omega} + \Sigma q h_i(\bar{x}))],
\]

which contradicts Inequality 2.7. Part (i) is therefore true. A similar argument can be constructed to demonstrate part (ii).

\textbf{Proof of Proposition 13}

We first demonstrate parts (i) and (ii).

Let \( \{q^*_i(s)\}_{i=1}^n \) be a firm’s optimal hedging strategy and therefore a solution to Equation 2.24. For any unbiased hedging instrument \( h(\bar{x}) \), we define a firm’s expected profits

\[
H(h) \equiv G(s)E[I(W_0 + \bar{\omega} + h(\bar{x}))] + (1 - G(s))E[I(W_0 + h(\bar{x}))]
\]

\[
= I(W_0) - cG(s)\sigma^2_\omega - c\sigma^2_{h(\bar{x})} - 2cG(s)\sigma^2_{\bar{\omega},h(\bar{x})}
\]

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Where the second line follows from observing that $I(W) = a + bW - cW^2$, and $Eh(\tilde{x}) = E\tilde{\omega} = 0$.

Consider an agent choosing a quantity of hedging instruments $z$, to maximize $H(\cdot)$. Thus, he chooses

$$z^* = \arg \max_z I(W_0) - cG(s)\sigma^2_\omega - cz^2\sigma^2_{h(\tilde{x})} - 2cG(s)z\sigma^2_{\tilde{\omega},h(\tilde{x})}.$$  

The first order condition is necessary and sufficient, and therefore,

$$-2cz^*\sigma^2_\omega - 2cG(s)\sigma^2_{\tilde{\omega},h(\tilde{x})} = 0$$

$$z^* = -\frac{G(s)\sigma^2_{\tilde{\omega},h(x)}}{\sigma^2_{h(x)}}.$$

Thus, the maximized value of $H$ is

$$H = I(W_0) - cG(s)\sigma^2_\omega - c\frac{G(s)^4\sigma^4_{\tilde{\omega},h(x)}}{\sigma^4_{h(x)}} + 2cG^2(s)\frac{\sigma^4_{\tilde{\omega},h(x)}}{\sigma^2_{h(x)}}$$

$$= I(W_0) - cG(s)\sigma^2_\omega + c\frac{G(s)^2\sigma^4_{\tilde{\omega},h(x)}}{\sigma^2_{h(x)}}$$

$$= I(W_0) - cG(s)\sigma^2_\omega + cG(s)^2\sigma^2_\omega\rho^2_{\tilde{\omega},h(x)};$$

where $\rho_{\tilde{\omega},h(x)}$ is the correlation coefficient between $\tilde{\omega}$ and $h(x)$. Observe that this maximum value of $H$ is increasing in $\rho_{\tilde{\omega},h(x)}$. By definition, $\theta(x)$ is the maximum correlation portfolio, and therefore the optimal hedge position is

$$\beta(s) \equiv s_j - c\sigma^2_\omega (1 - 2G(s_j)\rho^2).$$

Part (iii) and (iv) follow from arguments in the text.

Proof of Proposition 14

Because the signals are drawn from an iid distribution, equation (ii) of proposition 13 shows
that the bids are also drawn from an iid distribution. Therefore the expected sample variance equals the distribution variance of \( \beta (s_j) \). From equation \((ii)\) of proposition 13, it is straightforward to have for the distribution variance

\[
\text{var}(\beta) = \text{var}(s) + (2\rho^2 c\sigma^2)^2 \text{var}(G(s)) + (4\rho^2 c\sigma^2) \text{cov}(s, G(s))
\] (2.11)

In the case of NH, the corresponding distribution variance is simple the variance of the signal (or equivalently it is given by plugging \( \rho = 0 \) into equation 2.11.)

\[
\text{var}(\beta^{NH}) = \text{var}(s)
\] (2.12)

we now prove the following claim.

Claim 1: \( \text{cov}(s, G(s)) > 0 \).

To prove the claim, first note that \( G(s) = F^{N-1}(s) \) and \( \text{E}[G(s)] = \frac{1}{N} \). Then \( \text{cov}(s, G(s)) = \)

\[
\begin{align*}
\text{E}[sG(s)] - \text{E}[s]\text{E}[G(s)] \\
= \text{E}[sG(s)] - \frac{1}{N}\text{E}[s] \\
= \int_{\frac{s}{2}}^{\frac{s}{N}} sF^{N-1}(s) dF(s) - \frac{1}{N} \int_{\frac{s}{2}}^{\frac{s}{N}} sdF(s) \\
= \frac{1}{N} \left[ \int_{\frac{s}{2}}^{\frac{s}{N}} sdF^{N}(s) - \int_{\frac{s}{2}}^{\frac{s}{N}} sdF(s) \right]
\end{align*}
\]

Because the distribution \( F^N(s) \) first order stochastically dominates over \( F(s) \) (because \( F^N(s) \leq F(s) \)), the above expression is positive. Therefore \( \text{cov}(s, G(s)) > 0 \). This proves the claim.

We now compare equations 2.11 and 2.12. Making use of claim 1 and notice that \( \text{var}(G(s)) > 0 \), we have \( \text{var}(\beta) > \text{var}(\beta^{NH}) \). This gives that \( \text{E}[\Sigma_{\beta}] > \text{E}[\Sigma_{\beta^{NH}}] \).

**Proof of Proposition 15**

To facilitate the calculation, observe that, if \( s \) is drawn from \( F(\cdot) \), then

\[
\text{E}[G(s)] = \int_{\frac{s}{2}}^{\frac{s}{N}} F^{N-1}(s)f(s)ds
= \frac{1}{N}
\] (2.13)

and
\[ \mathbb{E}[G^2(s)] = \int_2^8 F^{2N-2}(s) f(s) ds = \frac{1}{2N-1} \]  

(2.14)

and

\[ \mathbb{E} \left[ \left( \sum_{i=1}^N G(s_i) \right)^2 \right] = \mathbb{E} \left[ \sum_{i=1}^N \left( G(s_i) \right)^2 \right] + \mathbb{E} \left[ \sum_{i \neq j} G(s_i) G(s_j) \right] 
= N \mathbb{E} \left[ G^2(s) \right] + N(N-1) \mathbb{E} \left[ G(s) \right] \mathbb{E} \left[ G(s) \right] 
= \frac{N}{2N-1} + \frac{N-1}{N} \]  

(2.15)

We then have for the sample variance of relative hedging positions:

\[ \Sigma_q = \frac{1}{N-1} \sum_{i=1}^N \left( -G(s_i) \rho \right)^2 - \frac{N}{N-1} \left( \frac{1}{N} \sum_{i=1}^N G(s_i) \rho \right)^2 \]

The expectation of the first term is

\[ \frac{1}{N-1} \mathbb{E} \left[ \sum_{i=1}^N \left( -G(s_i) \rho \right)^2 \right] = \rho^2 \mathbb{E} \left[ G^2(s) \right] 
= \frac{N}{N-1} \frac{1}{2N-1} \rho^2 \]

and for the second term

\[ \frac{1}{N(N-1)} \mathbb{E} \left[ \left( \sum_{i=1}^N G(s_i) \rho \right)^2 \right] = \frac{\rho^2}{N(N-1)} \mathbb{E} \left[ \left( \sum_{i=1}^N G(s_i) \right)^2 \right] 
= \frac{\rho^2}{(N-1)(2N-1)} + \frac{\rho^2}{N^2} \]

where Equation 2.15 was used. Combining both terms, we get \( \mathbb{E}[\Sigma_q] \) as claimed. 

Proof of Proposition 16
Let \( s_1 \) be the highest signal. Then

\[
E[G(s_1)] = \int_{\frac{1}{2}}^{s} F^{N-1}(s) dF^{N}(s) = N \int_{\frac{1}{2}}^{s} F^{2N-2}(s) dF(s) = \frac{N}{2N - 1}.
\tag{2.16}
\]

\[
 \Sigma_{\Delta W} = \frac{1}{N - 1} \sum_{i=1}^{N} (\Delta W(s_i))^2 - \frac{N}{N - 1} \left( \frac{1}{N} \sum_{i=1}^{N} \Delta W(s_i) \right)^2.
\tag{2.17}
\]

The expected value of the first term is

\[
\frac{1}{N - 1} E \left[ \sum_{i=1}^{N} (\Delta W(s_i))^2 \right] = \frac{\sigma_{\omega}^2}{N - 1} E \left[ G^2(s_1) \rho^2 + 1 - 2\rho^2 G(s_1) + \sum_{i=2}^{N} \rho^2 G^2(s_i) \right]
\]

\[
= \frac{\sigma_{\omega}^2}{N - 1} E \left[ \sum_{i=1}^{N} G^2(s_i) \right] + \frac{\sigma_{\omega}^2}{N - 1} - \frac{2\rho^2 \sigma_{\omega}^2}{N - 1} E[G(s_1)]
\]

\[
= \frac{N}{N - 1} \left( \frac{\sigma_{\omega}^2 \rho^2 E[G^2(s)]}{2N - 1} + \frac{\sigma_{\omega}^2}{N - 1} - \frac{2\rho^2 \sigma_{\omega}^2}{2N - 1} \right)
\]

\[
= \frac{\sigma_{\omega}^2}{N - 1} \left( 1 - \rho^2 \frac{N}{2N - 1} \right)
\]

where Equations 2.15 and 2.16 were used.

The expected value of the second term in Equation 2.17 is

\[
\frac{N}{N - 1} E \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \Delta W(s_i) \right)^2 \right] = \frac{1}{N (N - 1)} E \left[ (\tilde{\omega} - \sum_{i=1}^{N} G(s_j) \rho \theta(\bar{x}))^2 \right]
\]

\[
= \frac{\sigma_{\omega}^2}{N (N - 1)} E \left[ 1 + \rho^2 \left( \sum_{i=1}^{N} G(s_i) \right)^2 - 2\rho^2 \sum_{i=1}^{N} G(s_j) \right]
\]

\[
= \frac{\sigma_{\omega}^2}{N (N - 1)} \left[ 1 + \rho^2 \left( \frac{N}{2N - 1} + \frac{N - 1}{N} \right) - 2\rho^2 \right]
\]

\[
= \frac{\sigma_{\omega}^2}{N (N - 1)} \left( 1 - \rho^2 \left( \frac{N - 1}{2N - 1} + \frac{1}{N} \right) \right)
\]

Combining both terms in Equation 2.17, we get \( E[\Sigma_{\Delta W}] \) as claimed.
Further, the corresponding result for no hedging instruments is obtained from the general case by setting $\rho = 0$, and that for contingent hedging is obtained from that of no hedging instruments by replacing $\sigma^2_\omega$ with $\sigma^2_\omega (1 - \rho^2)$.

**Proof of Proposition 17**

If the firm wins the auction, then

$$E[\Delta W(s_1)\omega] = E\tilde{\omega}\omega - G(s_1)\rho E\tilde{\omega}\theta(\tilde{x}) = \sigma^2_\omega - G(s_1)\rho^2\sigma^2_\omega.$$  

If the firm loses the auction, then

$$E[\Delta W(s_j)\omega] = -G(s_j)\rho^2\sigma^2_\omega.$$  

The result follows.

**Proof of Proposition 18**

We use an argument similar to that used to prove the standard revenue equivalence principle and demonstrate that this implies an agent’s payoff is independent of the number of hedging vehicles.

We proceed by constructing an agent’s incentive compatibility constraint. Let $\pi(s, z)$ denote the expected utility of a bidder when his signal is $s$ but bids and hedges as type $z$. We compare $\pi(s, z)$ with $\pi(z, z)$, the expected utility of a bidder who has a signal $z$ and bids and hedges as type $z$.

Since the agent in both these cases has the same probability of winning: $G(z)$, and are bidding the same and hedging optimally, the only difference in the expected profits is in their private valuations. Therefore,

$$\pi(s, z) - \pi(z, z) = (s - z) G(z)$$

Taking a partial derivative with respect to $z$ and evaluating at $z = s$ yields:

$$\frac{\partial \pi(s, z)}{\partial z} \bigg|_{z=s} = \frac{d}{dz} \bigg|_{z=s} \pi(z, z) = -G(s).$$

The first order condition of the bidding strategy implies that $\frac{\partial \pi(s, z)}{\partial z} \bigg|_{z=s} = 0$. Thus,

$$\frac{d}{ds} \pi(s, s) = G(s).$$
Thus, integrating
\[
\pi (s, s) = \pi (\bar{s}, \bar{s}) + \int_{\bar{s}}^s G (z) \, dz. \tag{2.18}
\]

Since bidder with $\bar{s}$ has zero chance of winning the auction and holds zero hedging instruments, \(\pi (\bar{s}, \bar{s}) = I(W_0)\). Therefore,
\[
\pi (s, s) = I(W_0) + \int_{\bar{s}}^s G (z) \, dz. \tag{2.19}
\]

This is independent of the hedging vehicles available. Note that \(E[V (s)] = \pi (s, s)\), this establishes the proposition.

**Proof of Proposition 19**

Let \(s_1\) and \(s_2\) denote the highest and second highest signals respectively. Then the ex-post value for a firm with signal \(s_j\) is

\[
V(s_j) = \begin{cases} 
[\Delta_f + I(W_0)] + c\sigma_\omega^2(1 - 2G(s_2)\rho^2) + m\Delta W(s_1) - c(\Delta W(s_1))^2 & \text{wins} \\
 m\Delta W(s_j) - c(\Delta W(s_j))^2 + I(W_0) & \text{loses}.
\end{cases}
\]

where \(\Delta f \equiv s_1 - s_2\).

We first calculate the expectation \(E[V (s_j)]\) conditional on the signal \(s_j\). Note that \(E[\Delta W(s_j)] = 0\). Further, if firm \(j\) wins, or equivalently \(j = 1\), then \(E[(\Delta W(s_1))^2] = \sigma_\omega^2(1 + G^2(s_1)\rho^2 - 2G(s_1)\rho^2)\). Recall that \(G (s) = F^{N-1} (s)\), and the distribution of \(s_2\) conditional on the highest signal being \(s_1\) is \(\frac{1}{F^{N-1}(s_1)}F^{N-1}(s_2)\), we then have

\[
E [G (s_2) | s_1] = \int_{\bar{s}}^{s_1} F^{N-1}(s_2) \frac{1}{F^{N-1}(s_1)}d(F^{N-1}(s_2))
\]

\[
= \frac{N-1}{F^{N-1}(s_1)} \int_{\bar{s}}^{s_1} F^{2N-3}(s_2) \, dF (s_2)
\]

\[
= \frac{1}{2} F^{N-1}(s_1)
\]

\[
= \frac{1}{2} G (s_1).
\]
On the other hand, if firm $j$ loses, then

$$E \left[ (\Delta W(s_j))^2 \right] = \rho^2 \sigma^2 \omega G^2(s_j)$$

Combining the above relations, we get

$$E[V(s_j)] = \begin{cases} E[\Delta_f] s_1 + I(W_0) + c \sigma^2 \rho^2 (G(s_1) - G^2(s_1)) & \text{if firm } j \text{ wins} \\ -c \rho^2 \sigma^2 \omega^2 G^2(s_j) + I(W_0) & \text{if firm } j \text{ loses.} \end{cases} \quad (2.20)$$

To calculate the unconditional expectation, note that $E[E[\Delta_f] s_1] = E[\Delta_f]$, and

$$E[G(s_1)] = \int_\omega^s F^{N-1}(s_1) dF^N(s_1)$$
$$= N \int_\omega^s F^{2N-2}(s_1) dF(s_1)$$
$$= \frac{N}{2N-1}$$

and

$$E[G^2(s_1)] = \int_\omega^s F^{2N-2}(s_1) dF^N(s_1)$$
$$= N \int_\omega^s F^{3N-3}(s_1) dF(s_1)$$
$$= \frac{N}{3N-2}$$

This allows us to compute $E[V]$. Note that $E[V^{NH}]$ can be derived by plugging $\rho = 0$ into $E[V]$, we have $E[V] - E[V^{NH}]$ as in the proposition.

To get the corresponding value for the losing firm, let $L(\cdot)$ denote the distribution of $s_j$ conditional on $s_j$ not being the highest, we then have from Bayes’ rule:

$$dL(s_j) = \frac{N}{N^2-1} (1 - G(s_j)) dF(s_j)$$

where $1 - G(s_1)$ and $\frac{N-1}{N}$ are the conditional and unconditional probabilities of $s_j$ not being the highest respectively. We then have
\[
E_{s_j} \left[ G^2 (s_j) \mid s_j < s_1 \right] = \int_{s}^{\infty} F^{2N-2} (s_j) \frac{N}{N-1} (1 - F^{N-1} (s_j)) \, dF (s_j) \\
= \frac{N}{(2N-1)(3N-2)}
\]

Again note that \( E[V^{NH}] \) can be derived by plugging \( \rho = 0 \) into \( E[V] \), which gives us \( E[V] - E[V^{NH}] \) for the losing firm as in the proposition.

This completes the proposition.

\textbf{Proof of Proposition 20}

Let \( s_2 \) be the second highest signal. From Lemma 13, we have

\[
\beta (s_2) = s_2 - c\sigma_\omega^2 \left[ 1 - 2G (s_2) \rho^2 \right],
\]

and

\[
\beta^C (s_2) = s_2 - c\sigma_\omega^2 \left[ 1 - \rho^2 \right].
\]

By setting \( \rho = 0 \) in the above equation, we have

\[
\beta^{NH} (s_2) = s_2 - c\sigma_\omega^2
\]

Since \( \pi_s = E[\beta (s_2)] \), we have

\[
\pi^C_s - \pi^{NH}_s = c\sigma_\omega^2 \rho^2
\]

Further, let \( F_2 (s_2) \) denote the distribution of \( s_2 \), we then have:

\[
F_2 (s_2) = NF (s_2)^{N-1} (1 - F (s_2)) + F (s_2)^N \\
= NF (s_2)^{N-1} - (N - 1) F (s_2)^N
\]

and thus we have
\[ \pi_s - \pi_s^{NH} = \sigma^2 \omega \int_{\frac{1}{2}}^{s} 2G(s_2) \rho^2 dF_2(s_2) \]

\[ = \sigma^2 \omega \int_{\frac{1}{2}}^{s} 2F(s_2)^{N-1} \rho^2 N (N - 1) (F(s_2)^{N-2} - F(s_2)^{N-1}) dF(s_2) \]

\[ = \sigma^2 \omega \rho^2 \int_{0}^{1} 2x^{N-1} \rho^2 N (N - 1) (x^{N-2} - x^{N-1}) dx \]

\[ = \sigma^2 \omega \rho^2 \frac{N}{2N - 1} \geq 0 \]

we then have

\[ \pi^C_s - \pi_s = (\pi^C_s - \pi^{NH}_s) - (\pi_s - \pi^{NH}_s) \]

\[ = \sigma^2 \omega \rho^2 \frac{N - 1}{2N - 1} \geq 0 \]

\[ \blacksquare \]

### 2.8 Appendix: Existence

We characterize the existence of equilibrium under somewhat weaker assumptions than those presented in the text. Specifically, we assume that

**Assumption 9.**

(i) The contracts are costly, so that \( q_i E[h_i(\tilde{x})] \leq 0 \), for all \( i = 1, ..., n \).

(ii) The contracts are non-cancelable: If (i) is satisfied and \( P(\sum_{i=1}^{n} q_i h_i(\tilde{x}) = 0) = 1 \), then \( q_i = 0 \) for all \( i \).

(iii) The payoff to each firm’s growth opportunity is profitable, or \( I(W_0) \geq W_1 \) for all \( W_1 \).

(iv) The payoff to each firm’s growth opportunity is sufficiently concave, so that there exists \( W'_1 \neq \tilde{W}_1 \), so that \( I'(W'_1) \neq I'(\tilde{W}_1) \).

To characterize Bayesian Nash equilibrium, we first determine the optimal hedging strategy for a given probability of winning. Then, given the hedging strategy, we determine optimal bid.

**Lemma 17.** For any probability, \( p \), of winning the auction, there exists a unique set of finite hedging quantities \( \{q^*_i\}^{n}_{i=1} \in Q \) that solve

\[ \{q^*_i(p)\}^{n}_{i=1} = \arg \max_{\{q_i\}^{n}_{i=1} \in Q} \{ pE[I(W_0 + \tilde{\omega} + \Sigma h_i(\tilde{x})]) + (1-p)E[I(W_0 + \Sigma q_i h_i(\tilde{x}))] \} \]  \hspace{1cm} (2.22)
Proof of Lemma 17

We first establish the following claims before proceeding.

Claim 1: For any random variable \( \tilde{r} \), \( \mathbb{E}[\tilde{r}] = \text{Pr}(\tilde{r} < 0)\mathbb{E}[\tilde{r}|\tilde{r} < 0] + \text{Pr}(\tilde{r} > 0)\mathbb{E}[\tilde{r}|\tilde{r} > 0] \).

We have from the law of iterated expectations:

\[
\mathbb{E}[\tilde{r}] = \text{Pr}(\tilde{r} < 0)\mathbb{E}[\tilde{r}|\tilde{r} < 0] + \text{Pr}(\tilde{r} = 0)\mathbb{E}[\tilde{r}|\tilde{r} = 0] + \text{Pr}(\tilde{r} > 0)\mathbb{E}[\tilde{r}|\tilde{r} > 0]
\]

Note that the second term is zero, we readily have the claim.

Claim 2: For any \( \{q_i\}_{i=1}^n \in Q \) and \( \{q_i\}_{i=1}^n \neq 0 \), \( \text{Pr}(\Sigma q_i h_i(x) < 0) > 0 \)

We prove the claim by contradiction. Suppose that \( \text{Pr}(\Sigma q_i h_i(x) < 0) = 0 \). Then from Claim 1,

\[
\mathbb{E}[\Sigma q_i h_i(x)] = \text{Pr}(\Sigma q_i^* h_i(x) < 0)\mathbb{E}[\Sigma q_i h_i(x) | \Sigma q_i h_i(x) < 0] + \text{Pr}(\Sigma q_i^* h_i(x) > 0)\mathbb{E}[\Sigma q_i^* h_i(x) | \Sigma q_i^* h_i(x) > 0]
\]

Further, from part (i) of the Assumption, the above expression must be non-positive. This implies that \( \text{Pr}(\Sigma q_i h_i(x) > 0) = 0 \). However, under the earlier assumption that \( \text{Pr}(\Sigma q_i h_i(x) < 0) = 0 \), then we must have \( \text{Pr}(\Sigma q_i h_i(x) = 0) = 1 \), which contradicts part (ii) of the Assumption that the hedging instruments are non-cancelable.

Claim 3: Define \( q_m \equiv \max_{\{q_i\}_{i=1}^n \in C_n} \text{Pr}(\Sigma q_i h_i(x) < 0)\mathbb{E}[\Sigma q_i h_i(x) | \Sigma q_i h_i(x) < 0] \) where \( C_n \equiv \{\{q_i\}_{i=1}^n | \{q_i\}_{i=1}^n \in Q \) and \( \sum_{i=1}^n q_i^2 = 1\} \). Then \( q_m < 0 \).

Notice that the set \( C_n \) is compact, and the maximization argument is a continuous function of \( \{q_i\}_{i=1}^n \), therefore there exists \( \{q_i^*\}_{i=1}^n \in C_n \) such that

\[
q_m = \text{Pr}(\Sigma q_i^* h_i(x) < 0)\mathbb{E}[\Sigma q_i^* h_i(x) | \Sigma q_i^* h_i(x) < 0].
\]

Note that \( \text{Pr}(\Sigma q_i^* h_i(x) < 0) > 0 \) from Claim 2, and \( E[\Sigma q_i^* h_i(x) | \Sigma q_i^* h_i(x) < 0] < 0 \) by construction, therefore \( q_m < 0 \).

Claim 4: For any \( \{q_i\}_{i=1}^n \in Q \) and \( \sqrt{\sum_{i=1}^n q_i^2} = r \), we have \( \text{Pr}(\Sigma q_i h_i(x) < 0)\mathbb{E}[\Sigma q_i h_i(x) | \Sigma q_i h_i(x) < 0] \leq r q_m \).

This readily follows from Claim 3 and the observation that \( \Sigma q_i h_i(x) \) is linear in \( \{q_i\}_{i=1}^n \).

Claim 5: For any \( \epsilon \in (-\infty, \infty) \), there exists \( r^* > 0 \) such that \( E[I(W_0 + \tilde{\omega} + \Sigma q_i h_i(x))] < \epsilon \) for any \( \{q_i\}_{i=1}^n \in Q \) and \( \sqrt{\sum_{i=1}^n q_i^2} > r^* \).

From part (iii) of the Assumption, we can assume \( I'(W_b) > I'(W_a) > 0 \) without loss of generality. Concavity of \( I(\cdot) \) gives that \( I(w) \leq I'(W_b) w + a_1 \) and \( I(w) \leq I'(W_a) w + a_2 \) for all \( w \), where \( a_1 \) and \( a_2 \) are some constants.

Define \( \Delta \equiv I'(W_b) - I'(W_a) > 0 \) and \( W_m \equiv W_0 + \tilde{\omega} \) where \( \tilde{\omega} \) is the upper bound of \( \tilde{\omega} \).
Further, for \( \{q_i\}_{i=1}^n \in Q \), let \( \tilde{r} \) denote \( \Sigma_{i=1}^n q_i h_i (\tilde{x}) \) for notational convenience. We have

\[
E[I (W_0 + \tilde{\omega} + \tilde{r})] \\
\leq E[I (W_m + \tilde{r})] \\
= \Pr (\tilde{r} > 0) E[I (W_m + \tilde{r}) | \tilde{r} > 0] + \Pr (\tilde{r} < 0) E[I (W_m + \tilde{r}) | \tilde{r} < 0] \\
\leq \Pr (\tilde{r} > 0) E[I' (W_a) (\tilde{r} + W_m) + a_1 |\tilde{r} > 0] + \Pr (\tilde{r} < 0) E[I' (W_b) (\tilde{r} + W_m) + a_2 |\tilde{r} < 0] \\
= \Delta \Pr (\tilde{r} < 0) E[\tilde{r} |\tilde{r} < 0] + I' (W_a) E[\tilde{r}] + \Pr (\tilde{r} > 0) (I' (W_a) W_m + a_1) + \Pr (\tilde{r} < 0) (I' (W_b) W_m + a_2) \\
\leq \Delta \Pr (\tilde{r} < 0) E[\tilde{r} |\tilde{r} < 0] + a_3 \\
\leq \Delta q_m r + a_3
\]

where \( r \equiv \sqrt{\sum_{i=1}^n q_i^2} \), \( a_3 \equiv I' (W_b) W_m + |a_1| + |a_2| \), and we have used Claims 1 and 4 and the fact that \( E[\tilde{r}] \leq 0 \). Since \( \Delta > 0 \) and \( q_m < 0 \), \( E[I (W_0 + \tilde{\omega} + \Sigma_{i=1}^n q_i h_i (\tilde{x}))] \) becomes arbitrarily small as \( r \) increases. This establishes the claim.

Claim 6: For any \( \epsilon \in (\infty, \infty) \), there exists \( r^* > 0 \) such that \( E[I (W_0 + \Sigma q_i h_i (\tilde{x}))] < \epsilon \) for any \( \{q_i\}_{i=1}^n \in Q \) and \( \sqrt{\sum_{i=1}^n q_i^2} > r^* \).

The proof is similar to that in Claim 5.

Claim 7: For any \( \epsilon \in (\infty, \infty) \), there exists \( r^* > 0 \) such that \( pE[I (W_0 + \tilde{\omega} + \Sigma q_i h_i (\tilde{x}))] + (1 - p) E[I (W_0 + \Sigma q_i h_i (\tilde{x}))] < \epsilon \) for any \( p \in [0, 1] \), any \( \{q_i\}_{i=1}^n \in Q \) and \( \sqrt{\sum_{i=1}^n q_i^2} > r^* \).

The proof follows directly from Claims 5 and 6 and is thus omitted.

We are now ready to prove the lemma. Utilizing Claim 7, there exists \( r^* > 0 \) such that

\[
pE[I (W_0 + \tilde{\omega} + \Sigma q_i h_i (\tilde{x}))] + (1 - p) E[I (W_0 + \Sigma q_i h_i (\tilde{x}))] < pE[I (W_0 + \tilde{\omega})] + (1 - p) I (W_0)
\]

(2.23)

for any \( \{q_i\}_{i=1}^n \in Q \) and \( \sqrt{\sum_{i=1}^n q_i^2} > r^* \). Further define a compact and convex space \( S_n \equiv \{\{q_i\}_{i=1}^n | \sqrt{\sum_{i=1}^n q_i} \leq r^*, \{q_i\}_{i=1}^n \in Q \} \). Notice that the argument in Equation 2.22 is continuous and concave in \( \{q_i\}_{i=1}^n \), therefore it obtains a unique maximum on \( S_n \) at some finite \( \{q_i^*\}_{i=1}^n \). Since \( \{q_i = 0\}_{i=1}^n \in S_n \), we have

\[
pE[I (W_0 + \tilde{\omega} + \Sigma q_i^* h_i (\tilde{x}))] + (1 - p) E[I (W_0 + \Sigma q_i^* h_i (\tilde{x}))] \geq pE[I (W_0 + \tilde{\omega})] + (1 - p) I (W_0)
\]

In light of Inequality 2.23, \( \{q_i^*\}_{i=1}^n \) is thus the unique solution for Equation 2.22 where the optimization is over the entire set \( Q \). This establishes the lemma.

Using Lemmas 16 and 15, we can show the existence of a unique increasing equilibrium. Specifically, each firm’s probability of winning the auction is the probability that it has the
highest bid, or that its private value to winning the auction is higher than the second highest: \( G(s_j) \).

**Proposition 21.** There exists a unique symmetric Bayesian Nash equilibrium in which the bidding strategy is increasing in \( s \). Let \( \{ q^n_i (s_j) \}_{i=1} \) and \( \beta(s_j) \) be a firm’s hedging and bidding strategies, then,

\[
(i) \{ q^n_i (s_j) \}_{i=1} = \arg\max_{\{ q^n_i \}_{i=1}} \begin{cases} 
G(s_j)E[I(W_0 + \tilde{\omega} + \Sigma q_i h_i(\tilde{x}))] + \\
(1 - G(s_j))E[I(W_0 + \Sigma q_i h_i(\tilde{x}))]
\end{cases} \tag{2.24}
\]

\[
(ii) \quad \beta(s_j) = s_j + E[I(W_0 + \tilde{\omega} + \Sigma q^n_i h_i(\tilde{x}))] - E[I(W_0 + \Sigma q^n_i h_i(\tilde{x}))]. \tag{2.25}
\]

**Proof of Proposition 21**

We first show that, if \( \{ q^n_i (s) \}_{i=1} \) and \( \beta(s) \) are a firm’s hedging and bidding strategies in a symmetric equilibrium and that \( \beta(s) \) is increasing, then equations (i) and (ii) must be satisfied. Note that, when a firm with signal \( s \) plays the equilibrium bidding strategy, the winning probability is then \( G(s) \), and this gives equation (i) through Lemma 17. Then Lemma 15 gives equation (ii).

Next we assume that \( \{ q^n_i (s) \}_{i=1} \) and \( \beta(s) \) are solutions to equations (i) and (ii), and we know from Lemma 17 that such solutions exist and are unique. Since \( G(s) \) increases in \( s \), we have from Lemma 16 that \( E[I(W_0 + \tilde{\omega} + \Sigma q_i h_i(\tilde{x}))] \) weakly increases in \( s \). Similarly \( E[I(W_0 + \Sigma q_i h_i(\tilde{x}))] \) weakly decreases in \( s \). Therefore \( \beta(s) \) strictly increases in \( s \). We now show that \( \{ q^n_i (s) \}_{i=1} \) and \( \beta(s) \) constitute an equilibrium.

We examine the best response for bidder 1 assuming all other bidders follow the strategies \( \{ q_i (s) \}_{i=1} \) and \( \beta(s) \). Assume bidder 1 has signal \( s \). Since \( \beta(s) \) strictly increases in \( s \) and there is no need to bid above \( \beta(\tilde{s}) \) or below \( \beta(\underline{s}) \), we assume she bids \( \beta(z) \) for some \( z \in [\underline{s}, \tilde{s}] \). Thus the winning probability is \( G(z) \). Following Lemma 17, the optimal hedging amounts are therefore \( \{ q^n_i (z) \}_{i=1} \) where \( \{ q^n_i (z) \}_{i=1} \) solves equation (i). Let \( \pi(s,z) \) be bidder 1’s expected profit when bidding as type \( z \) and hedging optimally, then

\[
\pi(s,z) = \int_{\underline{s}}^{z} (s - \beta(Y_1)) dG(Y_1) + G(z) E[I(W_0 + \tilde{\omega} + \Sigma q^n_i (z) h_i(\tilde{x}))] + \\
(1 - G(z)) E[I(W_0 + \Sigma q^n_i (z) h_i(\tilde{x}))]
\]

where \( Y_1 \) is the highest signal among the other \( N - 1 \) firms. Since \( \{ q^n_i (z) \} \) is optimal for a
bidder with winning probability $G(z)$, we apply the envelope theorem and have

$$\frac{\partial \pi (s, z)}{\partial z} = g(z) \left[ s - \beta(z) + E \left[ I (W_0 + \tilde{w} + \sum q_i^* (z) h_i (\tilde{x})) \right] - E \left[ I (W_0 + \sum q_i^* (z) h_i (\tilde{x})) \right] \right]$$

$$= g(z) (s - z)$$

where the last line follows from Equation 2.3. Since $\frac{\partial \pi (s, z)}{\partial z} > 0$ for $z < s$ and $\frac{\partial \pi (s, z)}{\partial z} < 0$ for $z > s$, $\pi (s, z)$ has a unique maximum at $z = s$, and therefore $\{q_i (s)\}_{i=1}^n$ and $\beta(s)$ are indeed the best response.

2.9 Appendix: Robustness

We have assumed that the payment does not affect the internal funds. To examine the robustness of this assumption, we consider conditions under which

$$E[I^A(W_0 + \tilde{w} + \sum_{i=1}^n q_i h_i (\tilde{x}) + s - \tilde{p})] \approx c_1 \left\{ E[I(W_0 + \tilde{w} + \sum_{i=1}^n q_i h_i (\tilde{x}))] + E\left[ s - \tilde{b} \right] \right\} + c_2 \ (2.26)$$

where $c_1, c_2$ are some constants, and $\tilde{b}$ is the random payment that the winner makes and is simply $\beta(s_2)$ conditional on $s_2 < s$, where $s_2$ is the second highest signal. If this approximation holds, then a model with growth function $I^A(\cdot)$ is qualitatively equivalent to a basic model with a growth function $I(\cdot)$. The form of $I(\cdot)$ may be different from $I^A(\cdot)$, even though we will see that they are similar. The conditions are:

**Assumption 10.** (i) $I^A(\cdot)$ is quadratic, or, $I^A(x) = a + bx + cx^2$.

(ii) $c_i \sigma_s << 1$, where $\sigma_s$ is the dispersion in the private benefit.

(iii) The hedging instruments are unbiased.

These assumptions are generally not restrictive. Assumption (i) assumes that the growth function is quadratic. As is discussed in Section 2.3, this is consistent with empirical evidence in Altinkilic and Hansen (2000). Furthermore, as long as the curvature of the growth function is sufficiently small, for example, if the marginal cost of raising capital is small, then Taylor expansion up to the quadratic term is a good approximation. Furthermore, the value of $c$ is small based on the findings in Atlinkilic (2000), and thus Assumption (ii) generally holds.

We first prove the following lemma.

**Lemma 18.** If $I(\cdot)$ is quadratic, then for any independent random variables $\tilde{x}$ and $\tilde{y}$ and constant $z$ such that $E[\tilde{x}] = 0$, we have $E[I (z + \tilde{x} + \tilde{y})] = E[I (z + \tilde{x})] + E[I (z + \tilde{y})] - I(z)$. 82
Proof. By direct calculation, we have
\[
E[I(z + \tilde{x} + \tilde{y})] = a + bz + bE[\tilde{y}] + cz^2 + cE[\tilde{x}^2] + 2zE[\tilde{y}]
\]
and
\[
88E[I(z + \tilde{x})] = a + bz + cz^2 + cE[\tilde{x}^2]
\]
and
\[
E[I(z + \tilde{y})] = a + bz + bE[\tilde{y}] + cz^2 + cE[\tilde{y}^2] + 2zE[\tilde{y}]
\]
Noting that \(I(z) = a + bz + cz^2\), the lemma can be readily verified.

We now establish the approximate equivalence between the two models.

**Proposition 22.** Under Assumption 10, the alternative model can be approximated by the basic model.

Proof. Notice that \(\tilde{b}\) is independent of \(\tilde{x}\) and \(\tilde{w}\), and therefore \((s - \tilde{b})\) is independent of \((\tilde{w} + \Sigma z_i h_i(\tilde{x}))\). Thus, if \(I^A(\cdot)\) is quadratic and hedging instruments are unbiased, we have from the lemma that,
\[
E[I^A(W_0 + \tilde{w} + \Sigma z_i h_i(\tilde{x}) + s - \tilde{b})]
\]
\[
= E[I^A(W_0 + \tilde{w} + \Sigma z_i h_i(\tilde{x}))] + E[I^A(W_0 + s - \tilde{b})] - I^A(W_0)
\]  (2.27)
\[
= E[I(W_0 + \tilde{w} + \Sigma z_i h_i(\tilde{x}))] + I^A(W_0)E[s - \tilde{b}] + cE[(s - \tilde{b})^2]
\]  (2.28)
\[
\approx E[I^A(W_0 + \tilde{w} + \Sigma z_i h_i(\tilde{x}))] + I^A(W_0)E[s - \tilde{b}]
\]  (2.29)
\[
= I^A(W_0) \left\{ E \left[ \frac{1}{I^{Av}(W_0)} I^A(W_0 + \tilde{w} + \Sigma z_i h_i(\tilde{x})) \right] + E[s - \tilde{b}] \right\}
\]  (2.30)
where the third line follows from Assumption (ii) and the fact that \(I^{Av}(W_0)\) is around one. Note that this assumption implies that \(cE[(s - \tilde{b})^2] << 1\), because \(s - \tilde{b}\) is the winner’s surplus, and is proportional to the dispersion in private benefit and inversely proportional to the number of bidders. Therefore the term \(cE[(s - \tilde{b})^2]\) is negligible compared with \(I^A(W_0)E[s - \tilde{b}]\) and thus can be safely neglected. We therefore have shown that the alternative model can be approximated by a basic model with growth function \(I(\cdot) = \frac{1}{I^{Av}(W_0)} I^A(\cdot)\).
Bibliography


Chapter 3

Fixed Revenue Auctions

This chapter is a joint work with Christine Parlour.

3.1 Introduction

A standard assumption in auction theory is that a seller has a unit (or many units) of a good which he wishes to sell at the highest possible price. Such “fixed quantity auctions” naturally apply to many situations; such as sales of art, wine, oil tracts, stamps or treasury bonds. However, frequently, a seller owning a large quantity seeks to sell the minimum required to raise a fixed amount. This description applies to an entrepreneur selling off shares in his company to raise a required investment amount; or to a firm in financial distress selling off assets in order to settle its most pressing obligations. Do the standard results and intuitions of fixed quantity auctions extend to these fixed revenue auctions?

To answer this question, we present a standard auction model based on Milgrom and Weber (1982). To avoid the collusive equilibria exhibited by Wilson (1979), in the fixed quantity auction (FQA), we restrict attention to bids that are not contingent on quantity. We interpret the bids submitted in the fixed revenue (FRA) case either as the price per unit of the good, or the quantity demanded of the good that the bidder is willing to accept in exchange for offering the fixed revenue. Further, in FQA, bidders receive signals on the cash value of the fixed quantity and they bid a certain cash payment in exchange for the fixed quantity; whereas in FRA, bidders have signals as to how many units of the good the fixed revenue is worth, and they bid a certain quantity.

We find that if the buyers’ valuations are private, a simple transformation on signals, allocations and payments allows us to express a fixed quantity auction as a fixed revenue one. Indeed,
the bidding strategies are similar, and the revenue equivalence principle (Riley and Samuelson 1981 and Myerson 1981) translates into a quantity equivalence principle, and the optimal mechanism design is also similar.

In the interdependent values case, however, such equivalence breaks down. Specifically, increasing symmetric equilibria may not exist in first price fixed revenue auction. This is because the quantity curve in FRA is downward sloping, giving bidders incentives to bid lower in first-price FRA. This effect combines with the winner’s curse effect, and may cause significant underbidding so that the bidder’s expected profit is no longer a concave function of the underlying value.

More importantly, the linkage principle is violated. As fixed revenue auctions naturally apply to financial transactions, this suggests that it is important to consider the difference between the two forms. The linkage principle, when applied to FQA, suggests that the seller’s expected revenue is larger if more information is released in the auction because information release mitigates the winner’s curse problem and encourages bidders to bid higher. Further, it predicts a preference ordering for the seller over English, second-price and first-price auctions. However, in the case of FQA in which the seller minimizes the expected quantity he sells, the seller’s preference ordering predicted by the linkage principle can be completely reversed. This is because releasing information has two opposite effects on the expected quantity sold. On one hand, releasing information still reduces the winner’s curse and benefits the seller (the expected quantity sold is less); on the other hand, releasing information introduces fluctuations in the value of the good and this increases the quantity sold due to Jensen’s inequality. These two effects influence the quantity that the seller sells in opposite ways, and we show that either effect may dominate.

The fact that there is a downward sloping quantity curve in FRA has several effects. It gives bidders incentives to bid lower and this makes the quantity weighted transaction price lower in FRA than in FQA. In the special case when the dispersion in bidder’s valuation is negligible compared with its mean value, the difference between the two types of auctions is reduced because the quantity curve in FRA becomes almost flat, and therefore we expect that in that case, FRA should retain the general properties of FQA, for example in the ranking between different auction mechanisms and the effects of public information. Finally, since the quantity awarded is no longer constant in FRA, there is complication arising from correlation between the quantity awarded and the unit valuation in FRA. We show this correlation introduces an additional factor in comparing the expected quantity the seller sells across different FRA forms. This allows us to generalize the linkage principle in the case of FRA.

To our knowledge, this is the first paper to explicitly characterize fixed revenue auctions and
compare them to the standard auction form for both private and interdependent values. However
the private values case in this paper is related to three existing papers. First, in an experiment,
Deck and Wilson (2004) derive bidding strategies for a fixed revenue auction in a special case of
private values.

Second, Hansen (1988) studies auctions of endogenous quantity in which several producers
compete for the right to sell to a market characterized by a downward sloping demand curve
and producers are assumed to have private information about marginal cost, and the prices and
gains from trade are compared between different auction mechanisms. We note that the setting
in the above study is similar to this paper if the demand curve is of the form $\frac{1}{p}$ where $p$ is the
unit price of the good.

Third, DeMarzo, Kremer and Skrzypacz (2005) study an auction in which bidders compete
for the rights to a project which requires an initial fixed amount of investment, and the bids
they place are in the form of securities from the project’s cash flow. When the security used in
bidding is equity, their situation is the same as in this paper. Not surprisingly, some of their
results are closely related to the results in this paper. For example, the revenue equivalence for
equity bidding in their paper can be deduced from our quantity equivalence principle.

A conceptual difference between their paper and ours is evident in our respective assumptions
on the seller’s profit. In Demarzo, Kremer and Skrzypacz (2005), the seller’s profit is a product of
the quantity retained and the winner’s value. This is because the project is run by the winning
bidder and thus the cash flow depends on the winner’s value. By contrast, in this paper the
seller’s profit is simply the quantity he retains. We therefore implicitly assume that the seller’s
valuation is independent of the buyers’. This assumption is reasonable if the project is run by
the seller (for example in the case of an entrepreneur raising money from venture capitalists who
enjoys private perquisites of control), or if the seller attaches a private value to the good (for
example in the case of a financially distressed firm selling off productive assets.) However, our
assumption is not reasonable in the absence of such frictions. On the other hand, we note that
the particular specification of the seller’s profit affects only the seller’s preference over different
auction forms and is irrelevant to most results in our paper.

The layout of the paper is as follows. We first specify the model in Section 3.2, then we
discuss FRA under a private values assumption in Section 3.3. Then we investigate the more
general case of interdependent valuations in Section 3.4. Section 3.5 concludes. All proofs are in
the appendix.
3.2 The Model

A risk neutral seller, who owns a divisible good of size 1, tries to sell as little of it as possible to \( N \) risk neutral buyers in order to raise a fixed revenue, also normalized to 1. Bidders receive signals that are informative about the value of the good. The signals, denoted by vector \( \mathbf{X} \) are (weakly) positively affiliated with a symmetric joint probability density function \( f(x_1, ..., x_N) \) which is continuous with full support on \([X, \bar{X}]^N\). Let \( m(x_i) \) (or \( M(x_i) \)) denote the probability (or cumulative) marginal distribution for the signal of any bidder \( i \).

We consider both interdependent and private values. In the general case of interdependent valuations, bidder \( i \)'s value per unit of the good depends on his own signal and all other bidders' signals. Specifically,

\[
v_i(\mathbf{X}) = u(X_i, \mathbf{X}_{-i}),
\]

where \( \mathbf{X}_{-i} \) is the vector of signals for all other bidders, and the function \( u \) is the same for all bidders and is increasing in all components and symmetric in the last \( N - 1 \) arguments. In the special case of private valuations, bidder \( i \)'s valuation depends only on his own signal. Without loss of generality, we assume a bidder’s signal is his valuation, or

\[
u_i(\mathbf{X}_i) = X_i.
\]

We further define \( \underline{v} \equiv u(\bar{X}, \mathbf{X}_{-i}) \) as the lowest possible valuation when all bidders receive the lowest signal \( \bar{X} \). We assume \( \underline{v} > 0 \) so that the fixed revenue is always be raised.

We restrict attention to auctions in which bidders do not submit demand schedules. That is, they do not submit bids that are conditional on their allocation. We therefore avoid complications with auctioning divisible goods identified in, for example, Wilson (1979). Let \( b_i \) denote the bid submitted by bidder \( i \).

In a fixed revenue auction, we interpret the bid as the price the bidder is willing to pay per unit of the good. In other words, if bidder \( i \) submits a bid \( b_i \), then he is asking for \( \frac{1}{b_i} \) units of the good in exchange for providing a revenue of 1.

In general, a sealed bid auction is characterized by a payment rule and an allocation rule. The payment rule, denoted by \( \theta(b_1, ..., b_N, i) \), specifies the payment bidder \( i \) makes. The allocation rule, denoted by \( \alpha(b_1, ..., b_N, i) \), specifies the quantity of the good bidder \( i \) receives.

**Definition 5.** A Fixed Revenue Auction (FRA) is one in which the total revenue sums up to 1, or,
\[ \Sigma_{i=1}^{N} \theta (b_1, ..., b_N, i) = 1 \text{ for all } b_1, ..., b_N. \] (3.1)

and a Fixed Quantity Auction (FQA), is one in which the total allocation sums to 1, or,

\[ \Sigma_{i=1}^{N} \alpha (b_1, ..., b_N, i) = 1 \text{ for all } b_1, ..., b_N. \] (3.2)

For example, the allocation and payment rules in a first-price FRA are:

\[ \alpha^{(I)}(b_1, ..., b_N, i) = \frac{1}{\max \{b_1, ..., b_N\}} \mathbf{1}_{\{i = \arg \max_j \{b_j\}\}} \]

and

\[ \theta^{(I)}(b_1, ..., b_N, i) = \mathbf{1}_{\{i = \arg \max_j \{b_j\}\}}. \]

Finally, let \( Y_1 \) denote the highest signal among the remaining \( N - 1 \) signals for any given bidder, and we use \( G(\cdot|x) \) and \( g(\cdot|x) \) to denote the c.d.f and p.d.f. of \( Y_1 \) conditional on \( X_i = x \). We further define the following expressions:

\[ v(x, y) \equiv \mathbb{E}[v_1|X_1 = x, Y_1 = y] \] (3.3)

and

\[ \hat{v}(x, y) \equiv \mathbb{E}[v_1|X_1 = x, Y_1 < y]. \] (3.4)

### 3.3 Private Value Auctions

In the private values case, we establish equivalence between a fixed revenue auction and a fixed quantity auction. We demonstrate this by examining the best response of bidder \( i \), in a sealed bid auction, taking others’ bidding strategies, denoted by \( \beta_{-i}(\cdot) \), as given. Bidder \( i \) maximizes his expected profit, and his best response is

\[ \arg \max_{b_i} \mathbb{E}[x_i \alpha(b_1, ..., b_N, i) - \theta(b_1, ..., b_N, i)|x_i, f(\cdot), \beta_{-i}(\cdot)]. \] (3.5)

Since his signal, \( x_i \), appears in the conditioning information set in the above expression, it can be treated as a constant and we can divide the terms inside the expectation by \( x_i \) and rewrite
agent $i$’s best response as:

$$
\arg \max_{b_i} \mathbb{E} \left[ \alpha(b_1, ..., b_N, i) - \frac{1}{x_i} \theta(b_1, ..., b_N, i) \right] x_i, f(\cdot), \beta_{-i}(\cdot)
$$

(3.6)

$$
= \arg \max_{b_i} \mathbb{E} \left[ \left( -\frac{1}{x_i} \right) \theta(b_1, ..., b_N, i) - \left( -\alpha(b_1, ..., b_N, i) \right) \right] x_i, f(\cdot), \beta_{-i}(\cdot).
$$

(3.7)

Clearly, bidder $i$’s best response in the original auction is the same as in a new one in which the allocation rule is $\theta(b_1, ..., b_N, i)$ and the payment rule is $-\alpha(b_1, ..., b_N, i)$, and his signal is replaced by $\left( -\frac{1}{x_i} \right)$. Referring to Equations 3.1 and 3.2, the above operation transforms a FRA into a FQA.

Formally, we define the dual transformation as follows.

**Definition 6.** A dual transformation of signals, allocation and payment rules is

- **signals**: $x_i^d \equiv -\frac{1}{x_i}$, $i = 1, ..., N$ (3.8)
- **allocation rule**: $\alpha^d(b_1, ..., b_N, i) \equiv \theta \left( -\frac{1}{b_1^d}, ..., -\frac{1}{b_N^d}, i \right)$, $i = 1, ..., N$ (3.9)
- **payment rule**: $\theta^d(b_1, ..., b_N, i) \equiv -\alpha \left( -\frac{1}{b_1^d}, ..., -\frac{1}{b_N^d}, i \right)$, $i = 1, ..., N$ (3.10)

From the signal transformation in equation 3.8, it is straightforward to derive the joint density distribution $f^d(x_1^d, ..., x_N^d)$ in the dual auction:

$$
f^d(x_1^d, ..., x_N^d) = \frac{1}{\left| \frac{\partial x}{\partial x^d} \right|} f(x_1, ..., x_N)
$$

$$
= \frac{1}{\prod_{i=1}^{N} \left( x_i^d \right)^2} f \left( -\frac{1}{x_1^d}, ..., -\frac{1}{x_N^d} \right),
$$

(3.11)

and the marginal distribution $m^d(\cdot)$ in the dual auction:

$$
m^d(x_i^d) = \frac{1}{(x_i^d)^2} m \left( -\frac{1}{x_i^d} \right), \quad i = 1, ..., N
$$

(3.12)

**Proposition 23.** For any sealed bid auction of direct mechanism under private valuation, the bidding strategy $\{\beta_i(\cdot)\}_{i=1}^{N}$ forms a Nash Equilibrium if and only if the transformed strategy...
\[ \{\beta^d_i(\cdot)\}_{i=1}^N \text{ forms a Nash Equilibrium in the dual auction, where} \]
\[
\beta^d_i(x_i^d) \equiv -\frac{1}{\beta_i(x_i)} = -\frac{1}{\beta_i(-\frac{1}{x_i^d})} \quad i = 1, \ldots, N. \tag{3.13}
\]

The dual transformation effectively transforms both the signal and the bids by \((-\frac{1}{x_i})\). This is because we can think of a FRA as an auction in which bidders receive signals on how many units of the good is worth the fixed revenue, and they bid for a certain quantity in exchange. The transformation has a negative sign to ensure that it is increasing and that the order of the bids is preserved. Therefore, the dual transformation conveniently transforms a first-price (second-price) FRA into a first-price (second-price) FQA and this allows for the standard results in FQA to be carried over to the case of FRA.

**Corollary 19.** The dual transformation transforms a first-price (second-price) FRA into a first-price (second-price) FQA.

The dual transformation established above is general in that it applies to any sealed bid auction with private values. Such valuations need not be independent or symmetric, nor the bidding strategies symmetric. The crucial requirement is that bidders’ valuations are private, such that bidder \(i\)’s valuation is simply \(x_i\) which can be treated as a constant and be factored out in Equation 3.5. On the other hand, when valuations are interdependent, bidder \(i\)’s valuation is \(u(X_i, X_{-i})\), and can no longer be treated as a constant and factored out. Therefore, the equivalence breaks down.

### 3.3.1 Comparing Fixed Quantity and Fixed Revenue Auctions with Private Values

To fix ideas, consider the symmetric and increasing bidding strategy in a first-price FQA of \(\beta_i^{(I)d}(x_i^d) = \mathbb{E}[Y^d | Y^d_1 < x_i^d] \)\(^1\). Using equation 3.8 and relation \(Y^d = -\frac{1}{Y_i}\), we have for the bidding strategy in first-price FRA:

\(^1\)See (Krishna 2002)
\[
\beta_i^I(x_i) = \frac{1}{\beta_i^d(x^d_i)} \\
= -\frac{1}{\mathbb{E}\left[-\frac{1}{Y_1}\big| Y_1 < x_i\right]} \\
= \frac{1}{\mathbb{E}\left[\frac{1}{Y_1}\big| Y_1 < x_i\right]} \quad (3.15)
\]

The bidding strategy in equation 3.15 can be rewritten in an intuitive way as \( \frac{1}{\beta_i^d(x_i)} = \mathbb{E}\left[\frac{1}{Y_1}\big| Y_1 < x_i\right] \): the awarded quantity conditional on winning is the expected value of the corresponding quantity for the runner-up. Similarly one can show that the bidding strategy in a second-price FRA is truthful: bidders bid their actual valuations.

One immediate consequence is that, the expected quantity that the seller has to sell is the same for the first-price and second-price FRA. Indeed, we will show next that the quantity equivalence holds in general. Recall that, under revenue equivalence, in FQA, the expected revenue to the seller is the same across all auction formats under some general conditions. The transformation between FQA and FRA characterized in proposition 23 implies that, the revenue equivalence in FQA becomes the quantity equivalence in FRA.

**Proposition 24.** The expected quantity the seller has to sell is the same for any symmetric and increasing equilibrium of any FRA if (i) the values are independently and identically distributed, the distribution is continuous with full support above zero; (ii) bidders are risk neutral and the total number of bidders is fixed; (iii) the bidder with the highest valuation is the only one who gets any quantity award and the quantity awarded is finite.

We also note that, under the conditions in proposition 24, the quantity weighted transaction price is the same for FRA of all formats, and it is also the same for FQA of all formats. This is because the quantity weighted transaction price is the expected revenue over the expected quantity that the seller sells. In the case of FRA, the revenue is a constant and the expected quantity the seller sells is the same across all formats; in the case of FQA, the quantity the seller sells is a constant and the expected revenue is the same across all formats.

**Proposition 25.** Under the conditions in proposition 24, the quantity weighted transaction price is the same for FRA of all formats, and it is lower than that in the corresponding FQA with the same signal structure of any format.
3.3.2 Optimal Mechanism

The equivalence extends to the optimal mechanism. To see this, we assume that the winner is not necessarily the one with the highest value, and in addition, there can be more than one bidder who is awarded any quantity, and we allow the total sum of revenue contributed by all bidders to be less than 1 which corresponds to a partial or complete shutdown. To implement the shutdown, we assume that there is an external institution which is willing to purchase any amount of the good at a low unit price of $v_0$ where $0 < v_0 < v$.

Again, we consider only direct mechanisms.\(^2\) Making use of proposition 23, the seller’s objective to minimize the expected quantity he sells in FRA can be expressed as follows:

\[
\arg \min_{\alpha^{d}(\cdot),p^{d}(\cdot)} \mathbb{E} \left[ \sum_{i=1}^{N} \alpha (b_1,\ldots,b_N,i) + \frac{1 - \sum_{i=1}^{N} \alpha(b_1,\ldots,b_N,i)}{v_0} \right]
\]

\[
= \arg \min_{\alpha^{d}(\cdot),p^{d}(\cdot)} \mathbb{E} \left[ - \sum_{i=1}^{N} b_i^d (b_1^d,\ldots,b_N^d,i) + \frac{1 - \sum_{i=1}^{N} \alpha^d(b_1^d,\ldots,b_N^d,i)}{v_0} \right]
\]

\[
= \arg \max_{\alpha^{d}(\cdot),p^{d}(\cdot)} \mathbb{E} \left[ \sum_{i=1}^{N} \alpha^d (b_1^d,\ldots,b_N^d,i) + 1 - \sum_{i=1}^{N} \alpha^d (b_1^d,\ldots,b_N^d,i) \left( -\frac{1}{v_0} \right) \right]
\]

we see that the seller’s objective to minimize the expected quantity he sells in the FRA is consistent with his objective to maximize the expected profit in the dual transformed FQA if we assign a reservation utility in the FQA for keeping the object as:

\[
x_0 = -\frac{1}{v_0}. \tag{3.16}
\]

Therefore the optimal mechanism in the dual transformed FQA corresponds to the optimal mechanism in the FRA.

**Proposition 26.** Suppose valuations are private and the seller can sell any amount of the good to an outside party at a unit price of $v_0$. Then an auction minimizes the expected quantity the seller sells in raising a fixed revenue if and only if the dual transformed auction maximizes the expected revenue the seller garners when he sells a fixed quantity and attaches a unit reservation utility of $-\frac{1}{v_0}$ to any unsold quantity.

The above proposition allows us to construct the optimal FRA from an optimal FQA. Recall that the optimal FQA involves the concept of virtual valuation. To proceed, we assume signals

\(^2\)Restricting to direct mechanisms in examining the optimal mechanisms is without loss of generality due to the revelation principle.
are independent for simplicity. The virtual valuation in the dual transformed FQA is defined to be
\[
\psi_i^d = x_i^d - \frac{1 - M^d (x_i^d)}{m^d (x_i^d)} = \frac{1}{x_i} - \frac{1 - M (x_i)}{(x_i)^2 m (x_i)}
\]
This motivates us to define a virtual quantity in the FRA as the negative of the virtual value:
\[
\psi_i (x_i) \equiv -\psi_i^d (x_i) = \frac{1}{x_i} + \frac{1 - M (x_i)}{(x_i)^2 m (x_i)}
\] (3.17)

To explicitly derive the optimal mechanism in the FRA, we assume the design problem is regular.

**Definition 7.** The design problem is regular if \(\psi^d (x^d)\) is increasing in \(x^d\), or equivalently if \(\psi_i (x_i)\) is decreasing in \(x_i\).

A sufficient condition for regularity is that the hazard rate \(\frac{m(x)}{1 - M(x)}\) is increasing in \(x\). For comparison, when the design problem is regular, the optimal FQA is such that the seller sells the entire quantity to the bidder with the highest virtual value. Specifically, the allocation rule of the optimal FQA is derived by Myerson (1981) as:
\[
\alpha_{opt}^d (b_1^d, \ldots, b_N^d, i) = \begin{cases} 
1 & \text{if } \psi_i^d (x_i^d) = \max_j \{\psi_j^d (x_j^d)\} \text{ and } \psi_i^d (x_i^d) > x_0 \\
0 & \text{otherwise}
\end{cases}
\] (3.18)

Based on proposition 26, the optimal FRA is such that the seller raises the entire revenue from the bidder with the lowest virtual quantity.

**Proposition 27.** If the design problem is regular, the optimal FRA has the following allocation rule:
\[
\alpha_{opt} (b_1, \ldots, b_N, i) = \begin{cases} 
1 & \text{if } \psi_i (x_i) = \min_j \{\psi_j (x_j)\} \text{ and } \psi_i (x_i) < \frac{1}{v_0} \\
0 & \text{otherwise}
\end{cases}
\]

If there is more than one bidder with the minimum \(\psi_i (x_i)\) and \(\psi_i (x_i) < \frac{1}{v_0}\), then \(\alpha_{opt} (b_1, \ldots, b_N, i)\) can be any positive number for these bidders as long as the sum is 1.

The same as in the optimal FQA, there are infinite ways to implement the optimal mechanism in proposition 27. One convenient way is to impose an optimal reserve price \(r^*\) in either first-price or second-price FRA, such that
\[
\psi_i^f (r^*) = \frac{1}{v_0}
\] (3.19)
3.4 Interdependent Value Auctions

When values are interdependent, then symmetric and increasing equilibria sometimes do not exist in first-price FRA, and the linkage principle breaks down. For simplicity, we only consider the four standard auction formats. As in the FQA, the second-price and English FRA are different because bidders’ signals in the English auction are revealed as they drop out. We will first investigate the English, second-price and first-price FRA, then we examine the linkage principle.

We first note, that bidding strategies in second-price and English FRA are identical to those in a corresponding FQA with identical signal structure. The proofs are the same as those for FQA in Krishna (2002). Specifically, symmetric equilibrium strategies in a second-price FRA are given by

\[ \beta^H (x) = v(x, x). \]

Symmetric equilibrium strategies in an English FRA are given by

\[ \beta^k (x, p_{k+1}, \ldots, p_N) = u(x, \ldots, x, x_{k+1}, \ldots, x_N) \]

where \( \beta^k \) is the bidder’s dropout price when \( k \) bidders remain active and bidders \( k + 1 \) through \( N \) have dropped out.

However, in general, bidding strategies in FRA are more complicated than those in FQA due to the sloping quantity curve, but this complication is much more significant in the case of first-price auctions than in second-price auctions. This is because in second-price auctions, the quantity awarded to a particular bidder is a step function of his bid in both FRA and FQA given others’ bids, whereas in first-price auctions, this is so only for FQA but not for FRA, causing considerable complications in the first-price FRA. In particular, symmetric and increasing pure strategy equilibria in first-price FRA may not exist as we will show below. To proceed, we first derive a necessary condition for a symmetric and increasing pure strategy.

Let \( \beta^I (x) \) denote the symmetric and increasing equilibrium strategy, and we perform a change of variable by defining \( Q^I (z) \equiv \frac{1}{\beta^I (z)} \). The variable \( Q^I (z) \) corresponds to the quantity demanded by the bidder and it is easier to work with. Assume all but one bidder follow \( \beta^I (\cdot) \) and let \( \Pi^I (z, x) \) be the bidder’s expected profit when his signal is \( x \) but bids \( \beta^I (z) \) instead. It is straightforward to have

\[ \Pi^I (z, x) = \int_0^z (Q^I (z) v(x, y) - 1) g(y|x) dy \]  

(3.20)
Taking derivative with respect to $z$ to get

$$
\frac{\partial \Pi^I(z, x)}{\partial z} = (Q^I(z)v(x, z) - 1)g(z|x) + \frac{d}{dz}Q^I(z) \int_0^z v(x, y)g(y|x)dy
$$

(3.21a)

$$
= (Q^I(z)v(x, z) - 1)g(z|x) + \frac{d}{dz}Q^I(z)G(z|x)\hat{v}(x, z)
$$

(3.21b)

First-order condition gives that

$$
\frac{d}{dx}Q^I(x) = \left(\frac{1}{\hat{v}(x, x)} - \frac{v(x, x)}{\hat{v}(x, x)}Q^I(x)\right)\frac{g(x|x)}{G(x|x)}
$$

(3.22)

It will be shown in the appendix that the above equation has a unique solution:

$$
Q^I(x) = \frac{1}{\beta^I(x)} = \int_x^x \frac{1}{v(y, y)}dL(y|x)
$$

(3.23)

where

$$
L(y|x) \equiv e^{-\int_x^y s(t)dt}
$$

(3.24)

and

$$
s(x) \equiv \frac{v(x, x)g(x|x)}{\hat{v}(x, x)G(x|x)}
$$

(3.25)

The function $L(\cdot|x)$ has the property that $L(X|x) = 0$ and $L(x|x) = 1$. Thus $L(\cdot|x)$ can be thought of as a cumulative distribution function on $[X, x]$.

Thus we have the following proposition:

**Proposition 28.** If a first-price FRA has a symmetric and increasing pure strategy equilibrium, then it is given by equation 3.23.

In the case of first-price FQA, the necessary condition for a symmetric and increasing equilibrium is also sufficient. In other words, a symmetry and increasing pure strategy equilibrium always exists in a first-price FQA. However, this is not true in FRA. The necessary condition in equation 3.23 is not a sufficient condition in general, and symmetric and increasing equilibria may not exist in FRA.

To gain intuition into why symmetric and increasing equilibria may not exist in first-price FRA, we assume that the equilibrium exists and consider $\Pi^I(z, x)$, a bidder’s expected profit when he has signal $x$ but follows the equilibrium strategy of an agent with signal $z$. The function
\( \Pi'(z, x) \) may not be concave in \( z \). Specifically, the downward sloping quantity curve and the winner’s curse effect both make bidders underbid in equilibrium (assume one exists). If the combined effect is strong enough so that underbidding is severe, then a bidder can benefit if he deviates and bids as if he has a higher signal. This is because underbidding reduces the cost of deviation since the bidder only has to increase his bid slightly; on the other hand, such a deviation can significantly benefit the bidder by increasing both the bidder’s winning probability and the expected value of the good conditional on winning. If the benefit outweighs the cost, such an equilibrium cannot be sustained.

We now construct a numerical example in which symmetric and increasing equilibria do not exist. Details of the calculation are presented in the appendix, and here we present the main results.

There are two bidders in the numerical example. To maximize the effect of winner’s curse, we assume a pure common value so that \( u = \frac{1}{2}x_1 + \frac{1}{2}x_2 + v \), where \( v > 0 \). Signals are independent with marginal distribution:

\[
    f(x) = \begin{cases} 
        \frac{\epsilon}{\Delta x} & \text{if } \Delta x < x < 1 \\
        \frac{1 - \epsilon (1 - \Delta x)}{\Delta x} & \text{if } 0 < x < \Delta x
    \end{cases}
\]

We proceed by assuming that symmetric and increasing equilibrium exists, then we examine the equilibrium bidding strategy \( \beta^I(x) \) and the payoff function \( \Pi^I(z, x) \) and show that the equilibrium is not valid.

Intuitively, since the supply is one over the price in FRA, the curvature of the supply curve as a function of the price is largest when the price is close to zero. Thus, if the distribution of the bidder’s valuation has a large component near zero, underbidding will be severe. This suggests that we choose \( \Delta x, \epsilon \) and \( v \) small. In this example, we choose \( \epsilon = 0.1 \) and \( v = 0.1 \), and we take the limit \( \Delta x \to 0 \) which simplifies the calculation.

We numerically calculate \( \beta^I(x) \) and \( \Pi^I(z, x) \). Figure 3.1 plots the bidding strategy \( \beta^I(x) \). We see that the bids are very low and \( \beta^I(1) = 0.114 \) which is only 0.014 above \( v \). The severe underbidding is purposely designed through the combination of the large concentration (90%) of the signal distribution at zero and the steeply downward sloping supply curve with a low \( v \).

We show that \( \beta^I(x) \) can not be the equilibrium strategy. Suppose bidder 2 follows \( \beta^I(\cdot) \) and bidder 1 has a signal \( x_1 = 0 \). If bidder 1 follows the equilibrium strategy by bidding \( \beta^I_r(0) = v = 0.1 \), then his expected profit is zero because his winning probability is zero. Now suppose he deviates and bids \( \beta^I_r(1) = 0.114 \) instead, then his winning probability is 1 and his expected profit...
profit is:

\[ \Pi^I(1, 0) = \frac{E[v|x_1 = 0, x_2 < 1]}{\beta^I(1)} - 1 = \frac{v + E[\frac{1}{2}x_2]}{0.114} - 1 = \frac{0.1 + \frac{1}{2} \times 0.1 \times 0.5}{0.114} - 1 = 0.096 \]

which is positive and is thus greater than the equilibrium profit.

Figure 3.2 plots the value of \( \Pi^I(z, x) \) as a function of \( z \) for \( x = 0 \). We see that \( \Pi^I(z, x) \) is not a concave function of \( z \). Even though \( z = 0 \) is still a local maximum, the function increases with \( z \) after an initial decrease, and it attains a maximum value of 0.096 at \( z = 1 \) which is consistent with our above calculation.

As a more dramatic example of the non-concavity of the payoff function, figure 3.3 plots the value of \( \Pi^I(z, x) \) as a function of \( z \) for \( x = 0.5 \). We see that \( z = 0.5 \) is only a local minimum. When \( z \) decreases from 0.5, \( \Pi^I(z, x) \) first increases and then decreases because \( \Pi^I(0, x) = 0 \); when \( z \) increases from 0.5, \( \Pi^I(z, x) \) keeps increasing up to \( z = 1 \).

When either \( \epsilon \) or \( v \) increases, underbidding is less and we may expect the equilibrium to recover. Figure 3.4 plots the value of \( \Pi^I(z, x) \) as a function of \( z \) for \( \epsilon = 0.5, v = 0.1 \) and \( x = 0.5 \); figure 3.5 plots the value of \( \Pi^I(z, x) \) as a function of \( z \) for \( \epsilon = 0.1, v = 0.8 \) and \( x = 0.5 \). We see that indeed in both cases \( \Pi^I(z, x) \) is maximum at \( z = x = 0.5 \).

To summarize, the combined effect of downward sloping quantity curve and the winner’s curse may result in severe underbidding and render symmetric and increasing equilibria non-existent in a first-price FRA. This is in contrast to a first-price FQA in which a symmetric and increasing equilibrium always exists.

With interdependent values neither the revenue equivalence principle in FQA nor the quantity equivalence principle in FRA holds. This means that the quantity weighted transaction price is no longer invariant across auction formats for either FQA or FRA. Therefore we need to specify the auction format in making the comparison. Specifically, we will examine the four standard auction formats. We show the same result as under private valuations that the quantity weighted transaction price is lower in FRA due to the downward sloping supply curve.

**Proposition 29.** The quantity weighted transaction price is lower for FRA than for the corresponding FQA with the same structure, for Dutch, English, second-price or first-price auctions (if a symmetric and increasing equilibrium exists in the first-price FRA).
Figure 3.1: Plot of $\beta_r^I(x)$ as a function of $x$ for $\epsilon = 0.1$ and $\underline{\nu} = 0.1$.

Figure 3.2: Plot of $\Pi_r^I(z, x)$ as a function of $z$ for $\epsilon = 0.1$, $\underline{\nu} = 0.1$ and $x = 0$. 
Figure 3.3: Plot of $\Pi(z, x)$ as a function of $z$ for $\epsilon = 0.1$, $v = 0.1$ and $x = 0.5$.

Figure 3.4: Plot of $\Pi(z, x)$ as a function of $z$ for $\epsilon = 0.5$, $v = 0.1$ and $x = 0.5$. 
Figure 3.5: Plot of $\Pi^I(z, x)$ as a function of $z$ for $\epsilon = 0.1$, $v = 0.8$ and $x = 0.5$.

3.4.1 The Linkage Principle

The linkage principle asserts that the seller is better off if more information is released in the auction because this minimizes the winner’s curse and encourages the bidders to bid higher. The linkage principle has two major implications. First, it implies that the seller’s expected revenue in an English FQA is (weakly) greater than that in a second-price FQA, which is still (weakly) greater than that in a first-price auction. Second, if the seller also has information concerning the value of the good, then the expected revenue to the seller is (weakly) larger if he always releases his information than always concealing it.

In the case of FRA, the seller minimizes the expected quantity he sells. We show that the linkage principle breaks down in all aspects in that the seller’s preference over different auction forms can be completely reversed from that predicted in FQA. The reason is that, releasing information in FRA has two competing effects on the expected quantity sold. On one hand, as in FQA, releasing information reduces the winner’s curse effect and benefits the seller (the expected quantity sold is less); on the other hand, releasing information introduces fluctuations in the value of the good and this increases the quantity sold due to Jensen’s inequality because quantity demanded is one over the value. These two effects influence the quantity sold by the seller in opposite ways, and either effect may dominate.
We first note that if the dispersion in bidders’ valuations is negligible compared with the mean value, then differences between FRA and FQA diminish because then the quantity curve in the FRA becomes almost flat. Thus, FRA and the corresponding FQA with the same signal structure will become similar. First of all, a unique symmetric and increasing equilibrium exists in a first-price FRA and the bidding strategy approaches that in the corresponding first-price FQA.

**Lemma 20.** Fix $u(\cdot) - \nu$ and let $\nu \to \infty$, then a unique symmetric and increasing pure strategy equilibrium exists in first-price FRA. The bidding strategy is given by equation 3.23 and is: $\beta^I(x) = \beta^I_q(x) + o\left(\frac{1}{\nu}\right)$ where $\beta^I_q(x)$ is the equilibrium bidding strategy in a corresponding first-price FQA with the same signal structure.

Secondly, the seller has the same preference among different forms of FRA as in FQA.

**Proposition 30.** Consider English, second-price or first-price FRA. If there are three or more bidders and bidder signals have strict positive affiliation, and if we fix $u(\cdot) - \nu$ and let $\nu \to \infty$, then the expected quantity sold in English auction is less than that in second-price auction, which is still less than that in first-price auction.

Next we show that there are situations where the ranking can be completely reversed.

**Proposition 31.** If there are three or more bidders and signals are iid, then the expected quantity sold in English auction is greater than that in second-price auction, which is still greater than that in first-price auction.

**Public Information**

We first enhance the symmetric model to include the seller’s information. Specifically, let the random variable $S \in [\underline{S}, \overline{S}]$ denote seller’s information which is positively affiliated with bidder’s signals $X$. Furthermore, we assume that bidder $i$’s valuation per unit of the good is

$$v_i(S, X) = u(S, X_i, X_{-i})$$

where the function $u$ is the same for all bidders and is increasing in all components and symmetric in the last $N - 1$ components. And we assume $\nu \equiv u(S, X, X_{-i}) > 0$.

We next show that, when the dispersion in bidder’s valuation is negligible compared with its mean value, the seller is better off by revealing his information in FRA, the same as in FQA.
Proposition 32. For English, second-price or first-price FRA, if seller’s signal and bidder signals have strict positive affiliation, and if we fix \( u(\cdot) = v \) and let \( v \to \infty \), then the expected quantity the seller sells is less if he always reveals his information than always hiding it.

Next we show that the comparison can go the other way.

Proposition 33. For English, second-price or first-price FRA, under the special case when bidder’s valuation only depends on seller’s signal, i.e., \( u(S, X_1, X_{-i}) = u(S) \), as long as \( u(S) \) is not a constant, then the expected quantity the seller sells is greater if he always reveals his information than always hiding it.

### 3.4.2 The Generalized Linkage Principle for FRA

The bidder in FRA maximizes the expected product of the quantity awarded and the unit valuation. Since the quantity awarded upon winning is not constant, the bidder is not only concerned with the expected quantity awarded, but also with the covariance between the quantity awarded and the unit valuation. This covariance represents an additional factor in comparing the expected quantity the seller sells across different auction forms. Namely, if an auction form gives bidder a larger covariance, the bidder will in turn demand a smaller expected quantity. We use this observation to establish a generalized linkage principle, or conditions under which releasing a seller’s information decreases the quantity that he has to give away.

Consider the symmetric and increasing equilibrium of FQA in which the highest bidder wins and pays a positive amount. Let \( W(z, x) \) denote the expected price paid by the winning bidder if he receives a signal \( x \) but bids as if his signal were \( z \). Let \( W_2(z, x) = \frac{\partial}{\partial x} W(z, x) \). In this context, the linkage principle in FQA is the following: Let \( A \) and \( B \) be two FQA in which the highest bidder wins and only he pays a positive amount, and suppose that each has a symmetric and increasing equilibrium. If (i) for all \( x \), \( W_2^A(x, x) \geq W_2^B(x, x) \); (ii) \( W^A(0, 0) = W^B(0, 0) \). Then the expected revenue in \( A \) is at least as large as the expected revenue in \( B \).

To generalize the linkage principle to FRA, we again consider the symmetric and increasing equilibrium in which the highest bidder wins and only he pays a positive amount. Let the random variable \( \tilde{Q}(z, x) \) denote the quantity awarded to the winner (which is assumed to be bidder 1 for convenience in the following discussions) if he receives a signal \( x \) but bids as if his signal were \( z \) and wins with that bid. Let \( Q(z, x) = \mathbb{E}[\tilde{Q}(z, x)] \) be the expected quantity awarded to the winner if he receives \( x \) but wins by bidding as if having a signal \( z \). We further define

\[
\text{cov}(z, x) \equiv \left\{ \mathbb{E}[\tilde{Q}(z, x) v_1 | X_1 = x, Y_1 < z] - Q(z, x) \hat{v}(x, z) \right\}
\]
and

\[ T(z, x) \equiv \text{cov}(z, x) G(z|x) \] (3.27)

where \( v_1 \) is bidder 1 (the winner)'s unit valuation and \( v(x, z) \) is defined in equation 3.4. We see that \( \text{cov}(z, x) \) is the conditional covariance between the awarded quantity and unit valuation, and \( T(z, x) \) is the winning probability weighted conditional covariance. For notational simplicity, let \( Q_2(z, x) \equiv \frac{\partial}{\partial z} Q(z, x) \) and \( T_1(z, x) \equiv \frac{\partial}{\partial z} T(z, x) \).

**Proposition 34. (generalized linkage principle in FRA)** Let \( A \) and \( B \) be two FRAs in which the highest bidder wins and only he pays a positive amount. Suppose that each has a symmetric and increasing equilibrium. If (i) for all \( x \), \( Q_2^A(x, x) \leq Q_2^B(x, x) \); (i) for all \( x \), \( T_1^A(x, x) \geq T_1^B(x, x) \); (iii) \( Q^A(0, 0) = Q^B(0, 0) \), then the expected quantity sold in \( A \) is at most as large as that in \( B \).

When valuations are private, proposition 34 implies the quantity equivalence principle as a special case. Suppose \( A \) and \( B \) are two FRA satisfying the conditions in proposition 24, we then have (i) for all \( x \), \( Q_2^A(x, x) = Q_2^B(x, x) = 0 \) because \( Q^A(z, x) \) and \( Q^B(z, x) \) only depend on \( z \); (ii) for all \( x \) and \( z \), \( T_1^A(z, x) = T_1^B(z, x) = 0 \) and thus \( T_1^A(x, x) = T_1^B(x, x) = 0 \) because \( v_1 \) depends only on \( x \) and thus is deterministic conditional on \( x \); (iii) \( Q^A(0, 0) = Q^B(0, 0) = \frac{1}{2} \).

And thus the expected quantity the seller sells is the same in \( A \) and \( B \).

In the more general case of interdependent valuations, proposition 34 shows that, the comparison between expected quantities sold in two forms of FRA is not only determined by \( Q_2(x, x) \), similar to the case of FQA, but also by an additional factor \( T_1(x, x) \), the marginal rate of change in the probability weighted covariance between the awarded quantity and the unit valuation. When \( T_1(x, x) \) is larger, the bidder tends to demand a smaller expected quantity in return. In many cases, these two factors tend to run in opposite directions which causes the ambiguities in comparing the expected quantity sold across different forms of FRA as discussed earlier.

For example, compare first-price and second-price FRA. We have \( Q_2^1(z, x) = 0 \) for all \( x, z \) because \( Q^I(z, x) \) is only a function of \( z \). We also have \( Q_2^{II}(z, x) \leq 0 \) because the larger \( x \) is, the more likely the second highest bid will also be larger (and hence a smaller awarded quantity) due to signal affiliation. Therefore we have \( Q_2^{II}(x, x) \leq Q_2^I(x, x) \). Furthermore, we have \( T_1^I(z, x) = 0 \) for all \( x, z \) because \( Q_2^I(z, x) \) is not random given the value of \( z \) and hence the term \( \text{cov}(z, x) \) vanishes and therefore \( T_1^I(z, x) = 0 \); on the other hand, we qualitatively have \( T_1^{II}(z, x) \leq 0 \) even though its sign can not be precisely determined. To see this, we have from equation 3.27:

\[ T_1(z, x) = \text{cov}(z, x) g(z|x) + \left[ \frac{\partial}{\partial z} \text{cov}(z, x) \right] G(z|x) \] (3.28)
When \( z = 0 \), we have \( \text{cov}^{II}(z,x) = 0 \) because \( \bar{Q}^{II}(0,x) = \frac{1}{v} \) which is not random; when \( z > 0 \), we have \( \text{cov}^{II}(z,x) < 0 \) due to negative correlation between \( \bar{Q}^{II}(z,x) \) and \( v_1 \): when \( \bar{Q}^{II}(z,x) \) is higher, the second highest signal is lower and hence \( v_1 \) is lower. Therefore we have \( \text{cov}^{II}(z,x) g(z|x) \leq 0 \) and qualitatively \( \frac{\partial}{\partial z} \text{cov}^{II}(z,x) \leq 0 \). Based on equation 3.28, we have \( T_1^{II}(z,x) \lesssim 0 \). As a result, we have \( T_1^{II}(z,x) \lesssim T_1^{I}(z,x) \). Consequently, the two factors \( Q_2(x,x) \) and \( T_1(x,x) \) have conflicting effects on the comparison of the expected quantity between second-price and first-price FRA. In the special case when \( v \) is large, the random variable \( \tilde{Q}(z,x) \) decreases as \( (v)^{-1} \) in its mean value, but its standard deviation decreases as \( (v)^{-2} \), and thus the \( \tilde{Q}_2(x,x) \) effect dominates over the \( T_1(z,x) \) effect, and that explains proposition 30; on the other hand when signals are independent, we then have \( Q_2^{II}(x,x) = Q_2^{I}(x,x) = 0 \) and hence only the effect of \( T_1(x,x) \) is important, and that qualitatively explains proposition 31.

### 3.5 Conclusion

In this paper we have investigated some general theoretical properties of FRA and compared them with the more familiar case of FQA. When buyers’ valuations are private, there is an equivalence between FRA and FQA and this gives rise to many interesting similarities. For example, the bidding strategies in English and second-price FRA are still truthful, and the bid in first-price FRA simply corresponds to the expected quantity of the runner-up. In addition, the revenue equivalence principle in FQA readily translates into quantity equivalence principle in FRA, and the concept of virtual valuation in FQA translates into that of virtual quantity in FRA in the optimal mechanism design.

However, FRA and FQA are different because of the downward sloping quantity curve in FRA. This difference has several implications. Firstly, the quantity weighted transaction prices are lower in FRA than in FQA. Secondly, symmetric and increasing pure strategy equilibria sometimes do not exist in first-price FRA when valuations are interdependent, and the non-existence occurs in situations where the downward sloping quantity curve and the winner’s curse effect combine to induce significant underbidding. For example, this can arise in the case of pure common values when the \( \frac{1}{p} \) quantity curve has a large curvature. In these cases, expected bidder profit is not concave in the underlying signal and equilibria fail to exist.

Thirdly, the linkage principle breaks down. On one hand, releasing information reduces the expected quantity the seller sells due to a reduction in the winner’s curse. On the other hand, it increases the expected quantity the seller sells through Jensen’s inequality because releasing information causes the valuation of the good to fluctuate. Finally, the correlation between the
awarded quantity and the unit valuation in FRA affects bidder’s expected profit and this induces an additional factor influencing the expected quantity the seller sells. When the marginal rate of change in the probability weighted covariance is larger, the bidder tends to demand a smaller expected quantity in return. This allows us to generalize the linkage principle to FRA in the quantity comparison across different auction forms.
Bibliography


3.6 Appendix: Proofs

Proof of Proposition 23

We will only prove the "only if" part, since the proof for the "if" part is the same. Assume strategy \( \{ \beta_i(x_i) \}^{N}_{i=1} \) forms a Nash Equilibrium in the original auction. That means, for any \( i \),

\[
\beta_i(x_i) = \arg \max_{b_i} \int \left[ x_i \alpha \left( \beta_1(x_1), \ldots, \beta_{i-1}(x_{i-1}), b_i, \beta_{i+1}(x_{i+1}), \ldots, \beta_N(x_N), i \right) - \theta \left( \beta_1(x_1), \ldots, \beta_{i-1}(x_{i-1}), b_i, \beta_{i+1}(x_{i+1}), \ldots, \beta_N(x_N), i \right) \right] f(x_1 \ldots x_N) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_N
\]

\[
= \arg \max_{b_i} \int \left[ \frac{1}{x_i} \alpha \left( \beta_1(x_1), \ldots, \beta_{i-1}(x_{i-1}), -\frac{1}{\beta_i^d}, \beta_{i+1}(x_{i+1}), \ldots, \beta_N(x_N), i \right) - \frac{1}{x_i} \theta \left( \beta_1(x_1), \ldots, \beta_{i-1}(x_{i-1}), -\frac{1}{\beta_i^d}, \beta_{i+1}(x_{i+1}), \ldots, \beta_N(x_N), i \right) \right] f(x_1 \ldots x_N) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_N
\]

where the second equation makes use of the fact that \( x_i > 0 \). We now perform a change of variable and replace all \( x_i \) with \( x_i^d \) as defined in equation 3.8. We can then rewrite the above equation as

\[
\beta_i(x_i) = \arg \max_{b_i} \int \frac{1}{x_i^d} \left[ x_i^d \alpha^d \left( \beta_1^d(x_1^d), \ldots, \beta_{i-1}^d(x_{i-1}^d), -\frac{1}{\beta_i^d}, \beta_{i+1}^d(x_{i+1}^d), \ldots, \beta_N^d(x_N^d), i \right) - \frac{1}{x_i^d} \theta^d \left( \beta_1^d(x_1^d), \ldots, \beta_{i-1}^d(x_{i-1}^d), -\frac{1}{\beta_i^d}, \beta_{i+1}^d(x_{i+1}^d), \ldots, \beta_N^d(x_N^d), i \right) \right] f^d(x_1^d \ldots x_N^d) dx_1^d \ldots dx_{i-1}^d dx_{i+1}^d \ldots dx_N^d
\]

\[
= \arg \max_{b_i} \int \frac{1}{\beta_i^d(x_i^d)}
\]

where we have used transformation in equation 3.13 (which implies \( \beta_i(x_i) = -\frac{1}{\beta_i^d(x_i^d)} \), among others) and made use of the relation that

\[
f(x_1 \ldots x_N) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_N = \frac{1}{(x_i^d)^2} f^d(x_1^d \ldots x_N^d) dx_1^d \ldots dx_{i-1}^d dx_{i+1}^d \ldots dx_N^d.
\]

We then have
\[ \beta^d_i (x^d_i) = \arg \max_{b_i} \int \left[ x^d_i \alpha^d_i \left( \beta^d_1 (x^d_1), \ldots, \beta^d_{i-1} (x^d_{i-1}), b_i, \beta^d_{i+1} (x^d_{i+1}), \ldots, \beta^d_N (x^d_N) \right), i - \theta \left( \beta^d_1 (x^d_1), \ldots, \beta^d_{i-1} (x^d_{i-1}), b_i, \beta^d_{i+1} (x^d_{i+1}), \ldots, \beta^d_N (x^d_N) \right), i \right] f^d(x^d_1 \ldots x^d_N) dx^d_1 \ldots dx^d_{i-1} dx^d_{i+1} \ldots dx^d_N \]

Since the above equation is true for all \( i \), we have that \( \{ \beta^d_i (x^d_i) \}_{i=1}^N \) forms a Nash Equilibrium in the dual auction. This completes the proof.

**Proof of Corollary 19**

Assume the original FRA is a first-price auction. Then we have for the corresponding FQA

\[ \alpha^{d(I)}(b^d_1, \ldots, b^d_N, i) = \theta^{d(I)} \left( -\frac{1}{b^d_1}, \ldots, -\frac{1}{b^d_N}, i \right) = 1 \left\{ i = \arg \max_{j} \left\{ -\frac{1}{b^d_j} \right\} \right\} = 1 \left\{ i = \arg \max_{j} \{ b^d_j \} \right\} \]

\[ \theta^{d(I)}(b^d_1, \ldots, b^d_N, i) = -\alpha^{d(I)} \left( -\frac{1}{b^d_1}, \ldots, -\frac{1}{b^d_N}, i \right) = -\frac{1}{\max \left\{ -\frac{1}{b^d_1}, \ldots, -\frac{1}{b^d_N} \right\}} 1 \left\{ i = \arg \max_{j} \left\{ -\frac{1}{b^d_j} \right\} \right\} = \max \left\{ b^d_1, \ldots, b^d_N \right\} 1 \left\{ i = \arg \max_{j} \{ b^d_j \} \right\} \]

Therefore the corresponding FQA is also first-price. The proof is similar for second-price.

**Proof of Proposition 24**

First note that, from proposition 23 and equations 3.10 and 3.13, for every bidder and every random outcome, his equilibrium allocation in the FRA is the negative of his equilibrium payment in the dual transformed FQA.

Next, note that the dual transformed FQA satisfies the following conditions: (i) the values are iid and the distribution is continuous; (ii) bidders are risk neutral and the total number of bidders is fixed; (iii) the bidder with the highest valuation is the only winner and the payment is finite. (iv) bidding strategies are symmetric and increasing. Therefore the revenue equivalence principle
applies such that bidder $i$’s expected payment in the dual transformed FQA conditional on having signal $x^d_i = -\frac{1}{x_i}$ is independent of the auction format. Therefore, his expected allocation in the FRA conditional on having a signal $x_i$ is independent of the auction format. This establishes the proposition.

**Proof of Proposition 25**

Let the random variables $\bar{p}$ and $\bar{q}$ denote the transaction price and quantity respectively, and let $\hat{p}$ denote the quantity weighted transaction price, we then have $\hat{p} = \frac{E[p_q]}{E[q]}$. For FRA, the numerator is the fixed revenue and the denominator is the same for all FRA due to quantity equivalence principle, therefore $\hat{p}_r$ is the same for all FRA; similarly for FQA, the denominator is the fixed quantity and the numerator is the same for all FQA due to revenue equivalence principle, therefore $\hat{p}_q$ is the same for all FQA. Consequently to compare $\hat{p}_r$ and $\hat{p}_q$, we can look at second-price auctions for convenience because the transaction price $\hat{p}$ is the same for both types of auctions due to truthful bidding. Using Jensen’s inequality, we have:

$$\hat{p}_r = \frac{1}{E[p]} < \frac{1}{E[p]} = E[p] = \hat{p}_q$$

This establishes the proposition.

**Proof of Proposition 26.**

The proof is given in the text.

**Proof of Proposition 27.**

The proof follows from proposition 26 and equation 3.18.

**Derivation of equation 3.23 and Proof of Proposition 28.**

To solve equation 3.22, multiply both side of equation 3.22 by $e^{\int_{X}^{x} s(t)dt}$ to get

$$\frac{d}{dx} \left[ e^{\int_{X}^{x} s(t)dt} Q^I (x) \right] = e^{\int_{X}^{x} s(t)dt} \frac{s(x)}{v(x,x)}$$

where $s(x)$ is defined in equation 3.25. The above equation gives upon integration

$$e^{\int_{X}^{x} s(t)dt} Q^I (x) = \int_{X}^{x} e^{\int_{X}^{y} s(t)dt} \frac{s(y)}{v(y,y)} dy + \frac{1}{v_l}$$
where we have used $Q^I(X) = \frac{1}{v}$. Therefore we have

$$Q^I(x) = \int_{X}^{x} e^{-\int_{x}^{s} s(t) dt} \frac{s(y)}{v(y,y)} dy + \frac{1}{v} e^{-\int_{x}^{X} s(t) dt}$$  \hspace{1cm} (3.29a)$$

$$= \int_{x}^{X} \frac{1}{v(y,y)} dL(y|x) + \frac{1}{v} L(X|x)$$  \hspace{1cm} (3.29b)$$

where $L(y|x)$ is defined in equation 3.24. We note that $L(\cdot|x)$ can be thought of as a distribution function on $[X,x]$. On one hand, we readily have $L(x|x) = e^0 = 1$; on the other hand we have $g(t|x)/G(t|x) \geq g(t|x)/G(t|X)$ for $t \geq X$ because of affiliation and $\frac{v(x,x)}{v(x,y)} \geq 1$, therefore

$$-\int_{X}^{x} s(t) dt \leq -\int_{X}^{x} \frac{g(t|X)}{G(t|X)} dt$$

$$= -\ln \frac{G(x|X)}{G(X|X)}$$

$$= -\infty$$

which gives that $L(X|x) = 0$. Therefore equation 3.29b becomes equation 3.23.

One subtlety in the above derivation is that, we have multiplied the term $e^{\int_{X}^{x} s(t) dt}$ to both sides of equation 3.22, but this term is infinite as we have just shown. However, we are still intact because we could have used a term $e^{\int_{X}^{x+\rho} s(t) dt}$ instead where $\rho$ is an arbitrary positive number and we will still get equation 3.23 as $\rho$ would cancel out.

This completes the proof.

**Calculation Details in the Numerical Example of Non-existence of Increasing Symmetric Equilibria in First-Price FRA.**

For $x > 0$ and $y > 0$ we have:

$$\hat{v}(t,t) = 0.5t + 0.25\epsilon t^2 / (1-\epsilon + \epsilon t) + v$$

and

$$L(y|x) = e^{-\int_{y}^{x} \frac{t^{1+v}}{0.5t + 0.25\epsilon t^2/(1-\epsilon + \epsilon t) + v}} 1^{1-\epsilon + \epsilon t} dt$$

Notice in the above expression, $L(0|x) > 0$ which seems to contradict our earlier assertion that $L(0|x) = 0$. This is because we have taken the limit $\Delta x \to 0$, and $L(\cdot|x)$ should abruptly drop to zero at zero. We will take care of this complication appropriately in the following calculation.
From equation 3.23 we have:

\[
Q^I_r(x) = \frac{1}{\beta^I(x)} = \int_0^x \frac{1}{v(y, y)} \left( e^{-\int_0^y 0.5 \xi + 0.25 \xi^2 / (1 - \epsilon + \epsilon z) + \frac{t + x}{y} - \epsilon \xi + \epsilon dt} \right) + \frac{1}{v(0, 0)} e^{-\int_0^y 0.5 \xi + 0.25 \xi^2 / (1 - \epsilon + \epsilon z) + \frac{t + x}{y} - \epsilon \xi + \epsilon dt}
\]

where the second term is the contribution from the discontinuity of \( L(\cdot | x) \) at zero. Then equation 3.20 becomes

\[
\Pi^I(z, x) = Q^I_r(z) \int_0^z v(x, y) f(y) dy - F(z)
\]

where \( Q^I_r(z) \) is the symmetric and increasing equilibrium in the corresponding first-price FQA with the same signal structure.

This completes the proof.

**Proof of Proposition 29.**

We first establish the following claim.

Claim: If a symmetric and increasing equilibrium exists in a first-price FRA, then \( \beta^I(x) \leq \beta^I_q(x) \) for all \( x \), where \( \beta^I_q(x) \) denotes the symmetric and increasing bidding strategy in the corresponding first-price FQA with the same signal structure.

To prove the claim, we first show that \( \beta^I(x) \leq \hat{\nu}(x, x) \). Equation 3.20 can be rewritten as

\[
\Pi^I(z, x) = \left( \frac{\hat{\nu}(x, z)}{\beta^I(z)} - 1 \right) G(z | x)
\]

and we have \( \Pi^I(x, x) = \max_z \Pi^I(z, x) \). Since \( \Pi^I(0, x) = 0 \), we have \( \Pi^I(x, x) \geq 0 \), and this gives \( \beta^I(x) \leq \hat{\nu}(x, x) \) due to equation 3.30. Since \( \hat{\nu}(x, x) \leq v(x, x) \), we also have \( \beta^I(x) \leq v(x, x) \).
Next we plug $Q^I(x) = \frac{1}{\beta(x)}$ into equation 3.22 to get:

\[
\frac{d}{dx} \beta^I(x) = \frac{\beta^I(x)}{v(x,x)} \left( v(x,x) - \beta^I(x) \right) \frac{g(x|x)}{G(x|x)} \leq \left( v(x,x) - \beta^I(x) \right) \frac{g(x|x)}{G(x|x)}
\] (3.31)

with the boundary condition of $\beta^I(X) = v$. From (4), the equation for $\beta_q^I(x)$ is

\[
\frac{d}{dx} \beta_q^I(x) = (v(x,x) - \beta_q^I(x)) \frac{g(x|x)}{G(x|x)}
\] (3.33)

with the same boundary condition $\beta_q^I(X) = v$. Therefore we have $\beta^I(X) = \beta_q^I(X)$ and $\frac{d}{dx} \beta^I(x) \leq \frac{d}{dx} \beta_q^I(x)$ for $x > 0$, and thus the claim is proven.

Next note that, for English and second-price auctions, the bidding strategy is the same for FRA and the corresponding FQA with the same signal structure. Therefore, if we let the random variables $\tilde{p}$ and $\tilde{p}_q$ denote the transaction prices for FRA and the corresponding FQA respectively, we have $\tilde{p} \leq \tilde{p}_q$ whether the auction format is first-price, second-price or English auction. Now let $\hat{p}$ and $\hat{p}_q$ denote the quantity weighted transaction prices for the FRA and the corresponding FQA respectively, we have:

\[
\hat{p} = \frac{1}{\text{E}[\frac{1}{\tilde{p}}]} \leq \frac{1}{\text{E}[\frac{1}{\tilde{p}_q}]} < \frac{1}{\text{E}[\tilde{p}]} = \text{E}[\tilde{p}_q] = \hat{p}_q
\]

Therefore, the proposition is proved.

\[\blacksquare\]

**Proof of Lemma 20**

Throughout the following proof, whenever we change $v$, we assume that $u(\cdot) - v$ is kept the same.

Claim 1: Under the limit that $v \to \infty$, the necessary condition for a symmetric and increasing bidding strategy in equation 3.23 satisfies the condition $\beta^I(x) = \beta_q^I(x) + O \left( \left( \frac{1}{v} \right) \right)$ for all $x \in [X, X]$, where $\beta_q^I(x)$ is the symmetric and increasing bidding strategy in the corresponding first-price FQA with the same signal structure.

To prove the claim, we make use of equation 3.22 to rewrite equation 3.21b as

\[
\frac{\partial \Pi^I(z, x)}{\partial z} = g(z|x) \left[ \left( \frac{\hat{v}(x,z) \frac{g(z|x)}{G(z|x)}}{\hat{v}(z,z) \frac{g(z|x)}{G(z|x)}} - 1 \right) + Q^I(z) v(x,z) \left( 1 - \frac{v(z,z) \frac{g(z|x)}{G(z|x)}}{\hat{v}(z,z) \frac{g(z|x)}{G(z|x)}} \right) \right]
\] (3.34)
First notice that the \( \frac{\partial \Pi^H(z,x)}{\partial z} \) = 0 when \( z = x \) which verifies the first-order condition.

We define \( \Delta \beta^I(x) \equiv \beta^I(x) - v \) and \( \Delta \beta^I_q(x) \equiv \beta^I_q(x) - v \). Since the bidding strategy in FQA shifts uniformly with the base valuation \( v \), \( \Delta \beta^I_q(x) \) is independent of \( v \), but \( \Delta \beta^I(x) \) is a function of \( v \) subjecting to the bound: \( 0 \leq \Delta \beta^I(x) \leq \Delta \beta^I_q(x) \) (refer to the claim in the proof of proposition 29). For all \( x, y \in [X, \bar{X}] \), define \( \Delta v(x, y) \equiv v(x, y) - v \) and \( \Delta \hat{v}(x, y) \equiv \hat{v}(x, y) - v \). We have that \( \Delta v(x, y) \) and \( \Delta \hat{v}(x, y) \) do not depend on \( v \). We use equation 3.31 and treat \( \frac{\Delta \beta^I(x)}{v} \) and \( \frac{\Delta \hat{v}(x, y)}{v} \) to be small quantities to get::

\[
\frac{d}{dx} \Delta \beta^I(x) = (\Delta v(x, x) - \Delta \beta^I(x)) \frac{g(x|x)}{G(x|x)} + o\left(\frac{1}{v}\right)
\]

Notice equation 3.35 is the same as the differential equation for \( \Delta \beta^I_q(x) \) expect for the \( o\left(\frac{1}{v}\right) \) term, and both \( \Delta \beta^I(x) \) and \( \Delta \beta^I_q(x) \) have the same boundary condition that \( \Delta \beta^I(X) = \Delta \beta^I_q(X) = 0 \), therefore we have proved the lemma.

Claim 2: Under the limit that \( v \to \infty \), equation 3.23 is sufficient for a symmetric and increasing strategy in the first-price FRA.

To prove the claim, we use claim 1 to reexamine equation 3.34 and have:

\[
Q^I(z) v(x, z) = \frac{v + \Delta v(x, z)}{v + \Delta \beta^I_q(z)} = 1 + \frac{\Delta v(x, z)}{v} - \frac{\Delta \beta^I_q(z)}{v} + o\left(\frac{1}{v^2}\right)
\]

and

\[
\frac{\hat{v}(x, z)}{\hat{v}(z, z)} = 1 + \frac{\Delta \hat{v}(x, z)}{v^2} - \frac{\Delta \hat{v}(z, z)}{v^2} + o\left(\frac{1}{v^2}\right)
\]

\[
\frac{v(x, z)}{\hat{v}(z, z)} = 1 + \frac{\Delta v(x, z)}{v^2} - \frac{\Delta \hat{v}(z, z)}{v^2} + o\left(\frac{1}{v^2}\right)
\]

\[
\frac{v(x, z)}{\hat{v}(x, z)} = 1 + \frac{\Delta v(x, z)}{v^2} - \frac{\Delta \hat{v}(x, z)}{v^2} + o\left(\frac{1}{v^2}\right)
\]

Plugging the above relations into equation 3.34 and we have:

\[
\frac{\partial \Pi^H(z, x)}{\partial z} = \frac{g(z|x)}{v} \left\{ [\Delta v(x, z) - \Delta \beta^I_q(z)] - [\Delta v(z, z) - \Delta \beta^I_q(z)] \frac{g(z|x)}{G(z|x)} \right\} + o\left(\frac{1}{v^2}\right)
\]

Note that we have \( \Delta v(z, z) \geq \Delta \beta^I_q(z) \) for all \( z \in [X, \bar{X}] \) by virtue of equation 3.33 and the fact that \( \frac{d}{dx} \beta^I_q(x) \geq 0 \). If \( z < x \), we further have \( \Delta v(x, z) > \Delta v(z, z) \) and \( \frac{g(z|x)}{G(z|x)} \leq 1 \) (due to
signal affiliation), and therefore we have \([\Delta v (x, z) - \Delta \beta_q^I (z)] - [\Delta v (z, z) - \Delta \beta_q^I (z)] \frac{\partial G(z)}{\partial (z)} \geq [\Delta v (x, z) - \Delta \beta_q^I (z)] - [\Delta v (z, z) - \Delta \beta_q^I (z)] > 0\); If \(z < x\), we similarly have \([\Delta v (x, z) - \Delta \beta_q^I (z)] - [\Delta v (z, z) - \Delta \beta_q^I (z)] \frac{\partial G(z)}{\partial (z)} < 0\). Therefore in the limit \(v \to \infty\) when \(o \left( \frac{1}{v^2} \right)\) can be safely ignored, we have \(\frac{\partial \Pi^I (z, x)}{\partial z} > 0\) if \(z < x\) and \(\frac{\partial \Pi^I (z, x)}{\partial z} < 0\) if \(z > x\), thus \(\Pi^I (z, x)\) obtains its maximum at \(z = x\).

Claim 2 establishes the lemma.

**Proof of Proposition 30**
Throughout the following proof, whenever we change \(v\), we assume that \(u (\cdot) - v\) is kept the same.

We prove the proposition by making use of lemma 20. Let \(E[\tilde{Q}^E], E[\tilde{Q}^II], \) and \(E[\tilde{Q}^I]\) denote the expected quantity sold in English, second-price, and first-price FRA respectively. Let random variable \(\tilde{p}^I\) be the transaction price in the first-price FRA. Let \(X_1\) denote the highest signal, we have \(\tilde{p}^I = \beta^I (X_1)\) and \(\tilde{p}^I_q = \beta^I_q (X_1)\), where subscript \(q\) denote variables associated with the corresponding FQA with the same signal structure. From lemma 20, we then have \(\tilde{p}^I = \tilde{p}^I_q + o \left( \frac{1}{v} \right) = v + \Delta \tilde{p}^I_q + o \left( \frac{1}{v} \right)\) where \(\Delta \tilde{p}^I_q \equiv \tilde{p}^I_q - v\) and \(\Delta \tilde{p}^I_q\) is independent of \(v\). Therefore

\[
E[\tilde{Q}^I] = E \left[ \frac{1}{\tilde{p}^I} \right] = E \left[ \frac{1}{v + \Delta \tilde{p}^I_q + o \left( \frac{1}{v} \right)} \right] = \frac{1}{v} E \left[ 1 - \frac{\Delta \tilde{p}^I_q}{v} + o \left( \frac{1}{v} \right) \right]
\]

similarly we have for second-price and English FRA:

\[
E[\tilde{Q}^II] = E \left[ \frac{1}{\tilde{p}^II} \right] = E \left[ \frac{1}{v + \Delta \tilde{p}^{II}_q} \right] = \frac{1}{v} \left\{ 1 - E \left[ \frac{\Delta \tilde{p}^{II}_q}{v} \right] + o \left( \frac{1}{v} \right) \right\}
\]

and

\[
E[\tilde{Q}^E] = E \left[ \frac{1}{\tilde{p}^E} \right] = E \left[ \frac{1}{v + \Delta \tilde{p}^E_q} \right] = \frac{1}{v} \left\{ 1 - E \left[ \frac{\Delta \tilde{p}^E_q}{v} \right] + o \left( \frac{1}{v} \right) \right\}
\]

Since we have \(E[\Delta \tilde{p}^E_q] > E[\Delta \tilde{p}^{II}_q] \geq E[\Delta \tilde{p}^I_q]\) under strict affiliation for three or more bidders, and the term \(o \left( \frac{1}{v^2} \right)\) can be ignored under the limit \(v \to \infty\), we have that \(E[\Delta \tilde{p}^E_q] > E[\Delta \tilde{p}^{II}_q] \geq E[\Delta \tilde{p}^I_q]\). 

**Proof of Proposition 31**
Throughout the following proof, whenever we change \(v\), we assume that \(u (\cdot) - v\) is kept the
same. Let $\tilde{E}^E$, $\tilde{E}^{II}$, and $\tilde{Q}^f$ denote the expected quantity sold in English, second-price, and first-price FRA respectively. We first show $\tilde{E}^E > \tilde{E}^{II}$. Let random variables $\tilde{p}^E$ and $\tilde{p}^{II}$ denote the transaction prices in English and second-price FRA respectively, we have $\tilde{p}^E = u (Y_1, Y_1, x_3, ..., x_N) + v$ and $\tilde{p}^{II} = v (Y_1, Y_1)$ where $x_3$ through $x_N$ refer to the third largest to smallest signals. Since signals are independent, we have:

$$E \left[ \tilde{p}^E | Y_1 = y \right] = E \left[ u (Y_1, Y_1, x_3, ..., x_N) + v | X_1 > y, Y_1 = y \right]$$  \hspace{1cm} (3.36)

$$= E \left[ u (Y_1, Y_1, x_3, ..., x_N) + v | X_1 = y, Y_1 = y \right]$$  \hspace{1cm} (3.37)

$$= v (y, y)$$  \hspace{1cm} (3.38)

$$= \tilde{p}^{II} (Y_1 = y)$$  \hspace{1cm} (3.39)

Notice that conditional on $Y_1 = y$, $\tilde{p}^E$ is still random but $\tilde{p}^{II}$ is deterministic, Jensen’s inequality and the law of iterated expectation give that

$$E \left[ \tilde{Q}^E \right] = E \left[ \frac{1}{\tilde{p}^E} \right]$$

$$= E \left[ E \left[ \frac{1}{\tilde{p}^E} | Y_1 = y \right] \right]$$

$$> E \left[ \frac{1}{E \left[ \tilde{p}^E | Y_1 = y \right]} \right]$$

$$= E \left[ \frac{1}{\tilde{p}^{II} (Y_1 = y)} \right]$$

$$= E \left[ \tilde{Q}^{II} \right]$$

Next we prove $\tilde{E}^{II} > \tilde{E}^f$. From equation 3.23 we have

$$E \left[ \tilde{Q}^f | X_1 = x, Y_1 < x \right] = \int_0^x \frac{1}{v (y, y)} dL (y | x)$$

On the other hand, we have

$$E \left[ \tilde{Q}^{II} | X_1 = x, Y_1 < x \right] = \int_0^x \frac{1}{v (y, y)} dK (y | x)$$

where $K (y | x) = \frac{F(y)}{F(x)}$ and $F (\cdot)$ is the signal distribution. Notice both $L (\cdot | x)$ and $K (\cdot | x)$ are distribution functions on $[X, x]$, we next show that $L (\cdot | x)$ first order stochastically dominates
over $K (\cdot | x)$. Using equation 3.25 and since $\frac{v(x,x)}{\bar{v}(x,x)} > 1$ for $x > 0$, we have that for $x > y > 0$:

$$- \int_y^x s(t) \, dt < - \int_y^x \frac{f(t)}{F(t)} \, dt = - \ln \frac{F(x)}{F(y)}$$

Then from equation 3.24 we have that $L (y|x) < \frac{F(y)}{F(x)} = K (y|x)$ for $x > y > X$, therefore $L (\cdot | x)$ first order stochastically dominates over $K (\cdot | x)$. Since $\frac{1}{\bar{v}(y,y)}$ is decreasing in $y$, we have

$$E[\tilde{Q}^{II} | X_1 = x, Y_1 < x] > E[\tilde{Q}^I | X_1 = x, Y_1 < x].$$

Using the law of iterated expectation, we have

$$E[\tilde{Q}^{II}] > E[\tilde{Q}^I].$$

Therefore the proposition is established.

**Proof of Proposition 32**

Throughout the following proof, whenever we change $\bar{v}$, we assume that $u (\cdot) - \bar{v}$ is kept the same.

We will only prove the statement for first-price auction, because the proof is similar for second-price and English auctions. Notationwise, we use subscript $q$ to denote variables associated with the corresponding FQA with the same signal structure, and we use subscript "reveal" and "hide" to denote the situations in which the seller always reveals or hides his information respectively. Now let random variables $\tilde{p}^I_{\text{reveal}}$ and $\tilde{p}^I_{q,\text{reveal}}$ be the transaction prices in the first-price FRA and the corresponding first-price FQA with the same signal structure respectively, if the seller always reveals the information. We first note that lemma 20 still holds when the seller also possesses information. We also note that the transaction price in a first-price auction is simply the bid of the bidder with the highest signal. Let the random variables $\tilde{p}^I$ and $\tilde{Q}^I$ denote the transaction price and the quantity the seller sells.

If the seller always reveals his information, we have

$$\tilde{p}^I_{\text{reveal}} = \tilde{p}^I_{q,\text{reveal}} + o \left( \frac{1}{\bar{v}} \right) = \bar{v} + \Delta \tilde{p}^I_{q,\text{reveal}} + o \left( \frac{1}{\bar{v}} \right).$$

Define $\Delta \tilde{p}^I_{q,\text{reveal}} \equiv \tilde{p}^I_{q,\text{reveal}} - \bar{v}$, then it is independent of $\bar{v}$. Therefore

$$E[\tilde{Q}^I_{\text{reveal}}] = E \left[ \frac{1}{\bar{v} + \Delta \tilde{p}^I_{q,\text{reveal}} + o \left( \frac{1}{\bar{v}} \right)} \right] = \frac{1}{\bar{v}} E \left[ 1 - \frac{\Delta \tilde{p}^I_{q,\text{reveal}}}{\bar{v}} + o \left( \frac{1}{\bar{v}} \right) \right]

= \frac{1}{\bar{v}} \left\{ 1 - E \left[ \frac{\Delta \tilde{p}^I_{q,\text{reveal}}}{\bar{v}} \right] + o \left( \frac{1}{\bar{v}} \right) \right\}$$
similarly when seller always hides the information, we have:

\[ E[\tilde{Q}_{r,\text{hide}}] = \frac{1}{\nu} \left( 1 - E\left[ \frac{\Delta \tilde{p}_{q,\text{hide}}}{\nu} \right] + o \left( \frac{1}{\nu} \right) \right) \]

Since we have \( E[\Delta \tilde{p}_{q,\text{reveal}}] > E[\Delta \tilde{p}_{q,\text{hide}}] \) under strict affiliation, and the term \( o \left( \frac{1}{\nu} \right) \) can be ignored under the limit \( \nu \to \infty \), the proposition is established.

**Proof of Proposition 33**

We use the same notation as in the proof of proposition 32. When the seller always hides his information, each bidder’s bid will be deterministic with \( \beta = E[u] \) and we have \( \tilde{Q}_{\text{hide}} = \frac{1}{E[u]} \). On the other hand, when seller’s information is always revealed, each bidder’s bid will be a function of \( S \) such that \( (S) = u(S) \), and we have \( \tilde{Q}_{\text{reveal}} = \frac{1}{u(S)} \). From Jensen’s inequality, we have

\[ E[h Q_{\text{reveal}}] > \frac{1}{E[u]} = \tilde{Q}_{\text{hide}}. \]

This completes the proof.

**Proof of Proposition 34**

Recall that \( \Pi (z, x) \) denote the expected profit to a bidder if he receives signal \( x \) but bids as if his signal were \( z \). We have

\[ \Pi (z, x) = G(z|x)E\left[ \tilde{Q} (z, x) v_1 - 1|X_1 = x, Y_1 < z \right] \]

\[ = G(z|x)E\left[ \tilde{Q} (z, x) v_1|X_1 = x, Y_1 < z \right] - G (z|x) \]

\[ = G(z|x)Q (z, x) \hat{v} (x, z) + T (z, x) - G (z|x) \]

first-order condition of \( \frac{\partial}{\partial z} \Pi (z, x) |_{z=x} = 0 \) gives

\[ g(x|x)Q (x, x) \hat{v} (x, x) + G(x|x)Q_1 (x, x) \hat{v} (x, x) + G(x|x)Q (x, x) \hat{v}_2 (x, x) + T_1 (x, x) - g (x|x) = 0 \]

(3.40)

where \( \hat{v}_2 (x, x) \equiv \frac{\partial}{\partial z} \hat{v} (x, z) |_{z=x} \). We now apply equation 3.40 to auctions A and B and define \( \Delta Q (x) \equiv Q^A (x, x) - Q^B (x, x) \) to get

\[ Q^A_1 (x, x) - Q^B_1 (x, x) = -\frac{[g (x|x) \hat{v} (x, x) + G(x|x)\hat{v}_2 (x, x)] \Delta Q (x) + [T^A_1 (x, x) - T^B_1 (x, x)]}{G(x|x)\hat{v} (x, x)} \]
Therefore we have

\[
\frac{d}{dx} \Delta Q(x) = [Q_A^1(x,x) - Q_B^1(x,x)] + [Q_A^2(x,x) - Q_B^2(x,x)]
\]

\[
= [Q_A^2(x,x) - Q_B^2(x,x)] - \frac{[g(x|x) \dot{v}(x,x) + G(x|x) \dot{v}_2(x,x)] \Delta Q(x) + [T_A^1(x,x) - T_B^1(x,x)]}{G(x|x) \dot{v}(x,x)}
\]

By hypothesis, we have \(Q_A^1(x,x) - Q_B^1(x,x) \leq 0\) and \(T_A^1(x,x) - T_B^1(x,x) \geq 0\). We also have \(g(x|x) \dot{v}(x,x) + G(x|x) \dot{v}_2(x,x) \geq 0\) because every term of it is non-negative. Therefore, whenever \(\Delta Q(x) \geq 0\) we must have \(\frac{d}{dx} \Delta Q(x) \leq 0\). Furthermore, \(\Delta Q(0) = 0\). Thus, we must have that for all \(x\), \(\Delta Q(x) \leq 0\). 

\[\blacksquare\]